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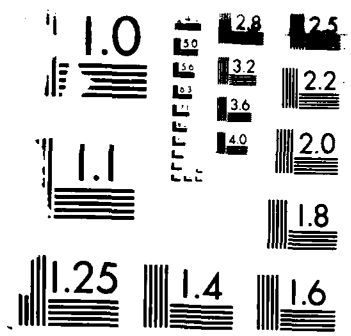


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IDA PAPER P-1902

# THE VALUE OF AREA DEFENSE IMPACT POINT PREDICTION IN A TWO LAYER DEFENSE WITH PERFECT ATTACKERS AND DEFENDERS

Michael V. Finn

September 1986



INSTITUTE FOR DEFENSE ANALYSES
1801 N. Beauregard Street, Alexandria, VA 22311



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A nationwide target data base is considered, containing targets of differing values. Each target is defended by a number of terminal interceptors corresponding to its value. Collections of targets are defended by area interceptors of longer range than the terminal interceptors. (The terminal interceptors are endoatmospheric and the area interceptors are exoatmospheric).

The paper determines the effect of preferential as opposed to random subtractive area defense. For any specified percent of target damage value, the number of RVs required in the presence and the absence of impact point prediction is found. For the cases examined more than twice as many RVs can be required if the defense has impact point prediction.

The value of maneuvering RVs to the attacker is the mirror image of the value of impact point prediction to the defender. If the RVs change direction after the exoatmospheric engagement, that engagement might as well have been random subtractive. Thus the paper addresses both the defender's value of having impact point prediction and the attacker's value of denying impact point prediction.

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IDA

INSTITUTE FOR DEFENSE ANALYSES

IDA Independent Research Program

## PREFACE

This study was conducted as part of the Independent Research Program of the Institute for Defense Analyses, under which significant issues of general interest to the defense research community are investigated.

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## I. DESCRIPTION OF THE PROBLEM

The offensive force consists of A identical, perfect attackers which we think of as RVs. The defense is a two-layer defense consisting of an area defense and terminal defense. All defensive weapons are perfect, so they destroy an incoming attacker if aimed at it. The targets have value v(i) for i=1,...,T, and we order the targets so that

$$v(1) \geq v(2) \geq \ldots \geq v(T).$$

The terminal defense is preallocated, with the number of interceptors at target i equal to v(i). The area defense can operate in two different modes - (1) <u>without impact point prediction and</u> (2) <u>with impact point prediction.</u> In the first case, no ipp, the area defense selects B of the A incoming RVs at random and fires interceptors at those RVs to kill them. (B is the number of area interceptors.) The RVs surviving continue to their respective targets, where they encounter the terminal defense. The terminal defense engages the RVs and if any RVs penetrate, the target is destroyed. In the second case, with ipp, the defense can see the structure of the attack and then, with this knowledge, allocate the area interceptors to specific RVs. For any attack, the total value destroyed will be greater in case (1) than in case (2), assuming that the defense employs the best strategy in the area defense layer. The problem is to find the optimal attacks in both cases (optimal meaning greatest expected value destroyed) and to examine quantitatively the difference between the two cases.

## II. A SIMPLE EXAMPLE AS A GUIDE TO UNDERSTANDING

In order to illustrate many concepts we will consider the following fairly simple situation:

- targets of value 10, 8, 5, 2, 1
- area defense, B=5
- number of attackers variable.

First consider the case with no ipp. Suppose the attacker has 10 weapons. His problem is to decide how to distribute the 10 weapons so as to maximize the expected damage, knowing that the defense does not have impact point prediction. Some points are clear:

- If the offense elects to attack target number i then it is useless to aim a number of RVs less than or equal to v(i). So in our case, the offense should not attack target 1 at all, if he attacks target 2 he should use at least 9 of his weapons, etc.

- If two targets, say target i and target j are attacked, then if $i < j$ (so that $v(i) \geq v(j)$), then an optimal attack must have

$$a(i) - v(i) \geq a(j) - v(j),$$

where a(k) is the number of attackers aimed at target k.

Professor Soland has written a dynamic programming algorithm to determine the optimal targeting for the attacker. Appendix A contains a short description of his algorithm.

Using his algorithm we find that with 10 weapons the optimal attack is

$$a(1) = 0$$

$$a(2) = 0$$

$$a(3) = 0$$

$$a(4) = 7$$

$$a(5) = 3,$$

and the expected damage is 2 1/3.

I have programmed the algorithm in FORTRAN on IDA's VAX 785 and run it for different numbers of attackers. A listing of the program and the outputs from its various runs can be found in Appendix A. The cases treated there are of area defense equal 5, 15, 25, 35, and 45.

The results are shown in Figure 1. The solid curves show the cases B=5, 15, 25. In each case the shape of the curve is similar, with a staircase at low attack levels and a smoother curve sloping to the right at higher attack levels.

Consider now the case where the area defense has impact point prediction. When the defense has ipp, the offense's optimal attack changes. There are two basic strategies that I have defined for the offense (1) shoot-to-kill and (2) defense dilution.

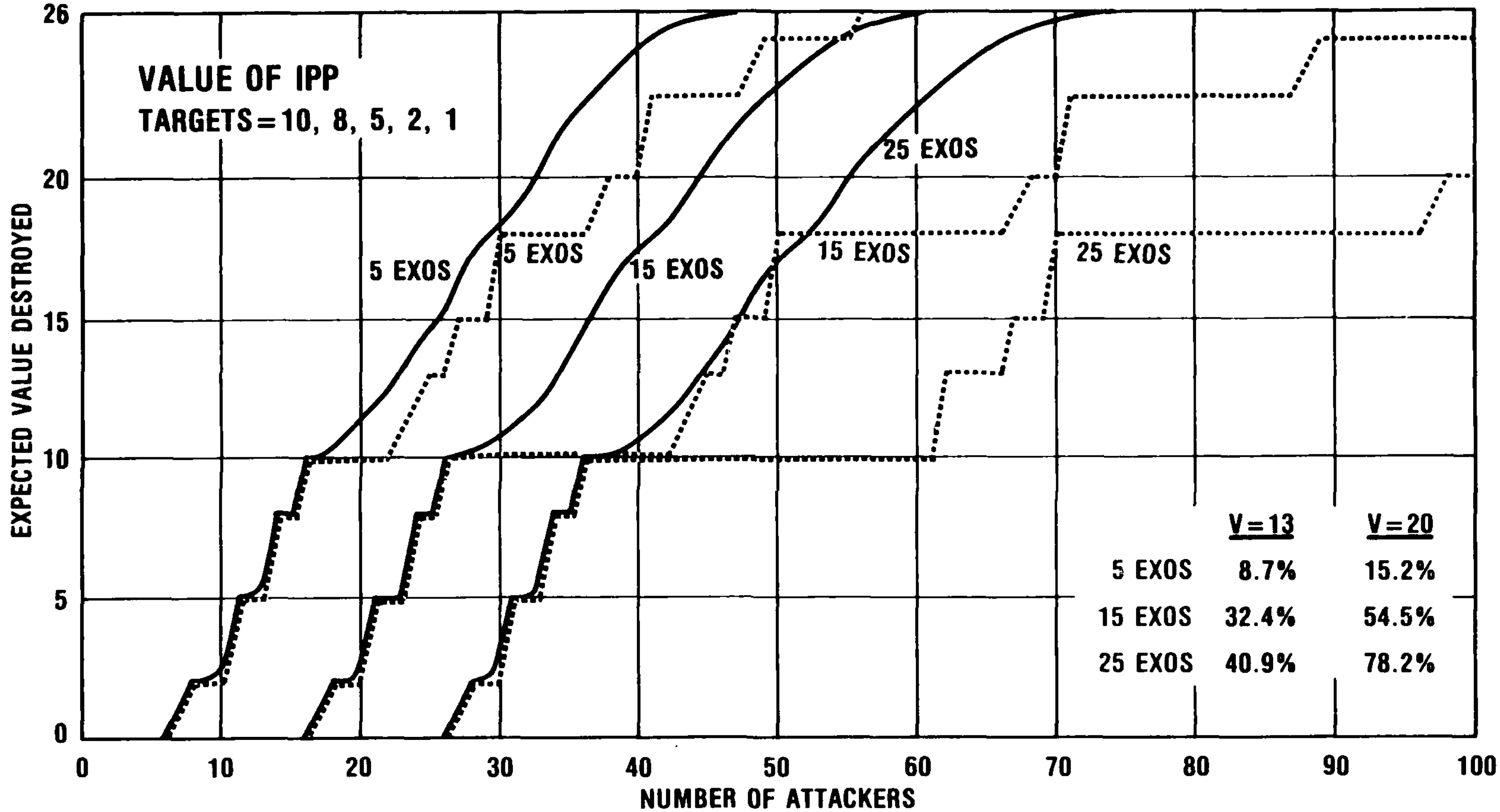


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Figure 1. Difference in the number of attackers needed to obtain a given damage level with and without impact point prediction in the case of five targets valued 10,8,5,2,1.

With the first strategy, the offense chooses a list of targets to attack and then attacks a target of value v with v+B+1 RVs. This shoot-to-kill attack effectively renders the area defense impotent to defend the target because even if the defense were to use all of his B area interceptors to defend the target, v+1 RVs will still penetrate to the terminal defense which consists of v interceptors -- the target is destroyed. Thus, at each attack level, the optimal shoot-to-kill strategy consists of picking the largest aggregate target value to attack subject to the condition of having enough RVs to attack a target of value v with v+B+1 RVs.

The maximum damage possible at each attack level for B=5, 15, 25, and 35 using the shoot-to-kill strategy can be found in Appendix B.

It turns out that the shoot-to-kill strategy is optimal for the offense when either the attack level is small and/or the number of area interceptors is small. In fact for B=5 and 15 discussed above, the shoot-to-kill strategy is optimal at all attack levels.

However, when B=25 (so we no longer have B "small") and A=62 (so the attack level is not "small") the shoot-to-kill strategy is not optimal. To prove it is not optimal, I shall demonstrate a better strategy. The optimal shoot-to-kill strategy would be to attack just the target of value 10 because no other feasible combination has a higher value destroyed. However consider the attack

$$a(1) = 10 + 13 = 23$$

$$a(2) = 8 + 13 = 21$$

$$a(3) = 5 + 13 = 18$$

$$a(4) = 0$$

$$a(5) = 0.$$

The defense, with 25 area interceptors, can save only one of the three targets attacked. To minimize his losses, he will save the value 10 target, so the offense destroys the targets of value 8 and 5 for a total damage of 13.

The offense has diluted the defense by attacking more targets than with the shoot-to-kill strategy. The offense does not dictate which targets will be destroyed with certainty but leaves that choice to the defense. What the offense can guarantee is the total damage.

Further properties of the defense dilution strategy will be examined in the next section.

By examining the possible attacks for the cases B=5, 15, 25 we can find the optimal attacks, as well as the expected damage. These results are plotted on Figure 1 as the dashed lines. Note that for a given number of area interceptors, the curve for ipp is always to the right and below the curve for no ipp.

The quantitative comparisons we will be making between the two curves will take the form:

> To achieve a specified damage level, what percentage more RVs must the offense expend when the defense has impact point prediction, as compared to the case when the defense does not have ipp?

For the cases B=5, 15, 25, and v=13, 20 the results are shown in the lower right-hand corner of Figure 1. For example, with 15 area interceptors the offense needs 54.5% more RVs to achieve a damage level of 20 when the defense has ipp than when it doesn't. Stated another way, if the offense can deny the defense impact point prediction (for example by maneuvering after the area defense layer) then the offense, to achieve a damage level of 20, can reduce his forces by approximately 1/3 (= .54/1.54).

## III. THE DEFENSE DILUTION STRATEGY

When the defense has many interceptors, the shoot-to-kill strategy is nonoptimal because the cost to attack a new target is high relative to its value. What is called for is a new strategy which we name the defense dilution strategy.

Suppose that the offense attacks all of the targets in the target set, meaning that a(i) > v(i) for all i. The area defense in his choice of which targets to defend (there is ipp) must solve a <u>knapsack problem</u>:

$$\max_{\substack{z_i \in \{0,1\} \\ \sum z_i (a(i)-v(i)) \leq B}} \sum z_i v(i).$$

This is generally a difficult problem to solve exactly. One heuristic solution would be to order the "densities"

$$\frac{v(i)}{a(i)-v(i)}$$

in increasing order, and then let

$$z_i = \begin{cases} 1 & i \le k \\ 0 & i > k \end{cases}$$

where

$$k = \underset{x}{\mathrm{MAX}} \text{ s.t. } \sum_{i=1}^{x} (a(i)-v(i)) \le B.$$

The value saved is the $\sum_{i=1}^{x} v(i)$. However, this leaves $B - \sum_{i=1}^{k} (a(i)-v(i))$ area interceptors unutilized, so that we suspect this is not optimal for the defense.

In fact, it is easy to make up examples where this strategy is not optimal. However, this strategy results in a value destroyed which is at most v(k+1) more than optimal, so if the target values are small relative to the total value, then the percentage difference between the two is small.

We can consider the continuous (or partitionable item) knapsack problem

$$\underset{z_i \,\varepsilon\, [0,1]}{\mathrm{MAX}} \quad \sum z_i \, v(i)$$
$$\sum z_i \, (a(i)-v(i)) \le B$$

which has the solution

$$z_i = \begin{cases} 1 & i \le k \\ \dfrac{B - \sum_{i=1}^{k} (a(i) - v(i))}{a(k+1) - v(k+1)} & i = k+1 \\ 0 & i \ge k+2. \end{cases}$$

The answer to this problem is at most v(k+1) <u>greater</u> than the original knapsack problem value. So again, if v(k+1) is small relative to the total value, the percentage difference is small.

The offense's problem is what might be called the "knapsack game":

$$\min_{\substack{\sum a(i)=A \\ a(i)>v(i)}} \quad \max_{\substack{z_i \in \{0,1\} \\ \sum z_i (a(i)-v(i)) \le B}} \quad \sum z_i \, v(i)$$

If the inside problem were the partitionable item knapsack problem ($z_i \; \varepsilon\{0,1\}$ instead of $z_i \, \varepsilon\{0,1\}$), which we assume is a good approximation, then the solution of the outside problem is to take a(i) such that

$$\frac{v(i)}{a(i) - v(i)} = \text{constant for each i},$$

i.e.,

$$a(i) = \frac{v(i)}{\sum v(i)} \bullet A \, .$$

(Note that we are letting a(i) be rational, but again this game is a good approximation to the game where a(i) is integral.)

This is what might be called the <u>Proportional Attack Theorem</u>. When the offense uses this attack, the value of the continuous game is

$$\begin{matrix}\text{VALUE} \\ \text{SAVED}\end{matrix} = \sum v(i) \frac{B}{A - \sum v(i)}$$

If we let $V = \sum v(i)$, then we have

$$\begin{matrix}\text{VALUE} \\ \text{DESTROYED}\end{matrix} = V(1 - \frac{B}{A-V})$$

Note however that at low attack levels the offense will not attack all of the targets (the above assumes that all of the targets are attacked). Let us assume that the offense will attack a fraction f of the total value. Then

$$g(f) = \begin{matrix}\text{VALUE} \\ \text{DESTROYED}\end{matrix} = fV \left\{1 - \frac{B}{A - fV}\right\}$$

The offense will pick f so that this is maximized. Taking the derivative,

$$\frac{dg}{df} = \frac{V}{(A-fV)^2}\left\{(A-fV)^2 - AB\right\}$$

so that at the maximum

$$fV = \sqrt{A}\,(\sqrt{A} - \sqrt{B}).$$

If $$V \le \sqrt{A}\,(\sqrt{A} - \sqrt{B})$$

then $$\begin{matrix}\text{VALUE} \\ \text{DESTROYED}\end{matrix} = (\sqrt{A} - \sqrt{B})^2$$

These formulas constitute the defense dilution strategy. For a more exact analysis of this knapsack game and the defense dilution strategy, see Appendix C. The algorithm there was never implemented - instead it was used as justification of the two formulas above for the expected value destroyed.

## IV. LARGE SCALE PROBLEMS WITH ONE AREA

In the process of solving the ipp vs. no ipp on a nationwide scale, we will encounter rather large size problems, in the number of targets, number of interceptors and number of attackers.

$$\text{The range of} \begin{cases} \text{number of targets is} & 43 - 80 \\ \text{total target value is} & 430 - 800 \\ \text{number area int is} & 43 - 800 \\ \text{number attackers is} & 50 - 2000. \end{cases}$$

As an example of the qualitative results for large scale problems compared to the small example in section II, I present the following situation

$$\text{targets} \begin{cases} \text{5 targets of value 20} \\ \text{10 targets of value 10} \\ \text{12 targets of value 5} \\ \text{Total value} = 260 \\ \text{Number of area interceptors} = 50{,}150{,}250. \end{cases}$$

The results are shown on Figure 2. As we see, the no ipp curves have become nearly lines and the ipp curves have smoothed quite a bit, except at the lower end where the shoot-to-kill strategy with its characteristic staircase shape is present.

The solid curves were calculated at 25 step intervals using Soland's algorithm and the dashed curves were calculated as a combination of stk and dd strategies.

I cannot account for the linear shape of the no ipp curves at present. However, they seem to be approximated by lines joining the following two points

$$(B,0) \text{ and } ((B+T+V)(1+\alpha), V)$$

where $T$ = number of targets and $\alpha$ is some number between .10 and .20.

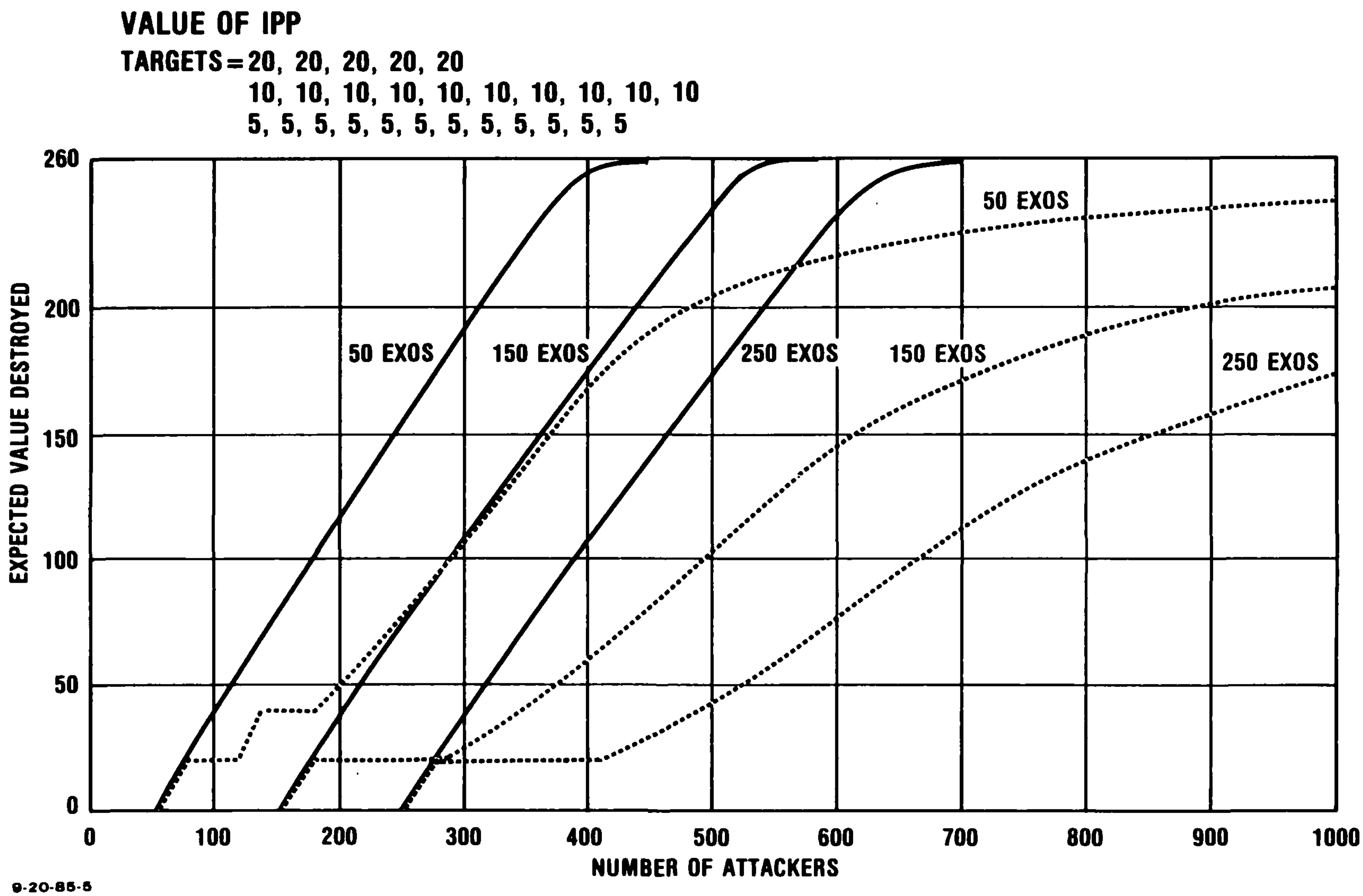
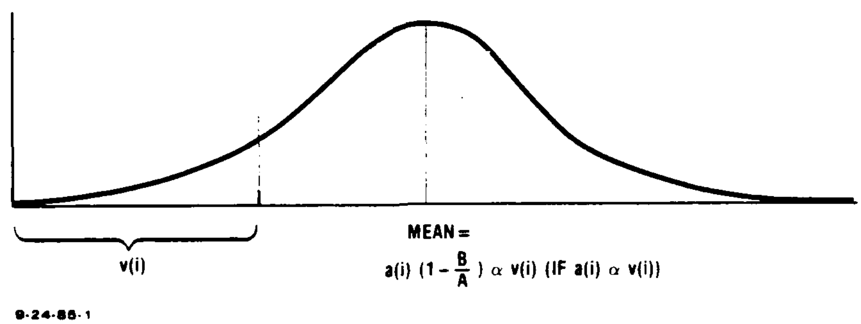



Figure 2. Difference between impact point prediction and no impact point prediction for a single citadel.

Perhaps there is some sort of Proportional Attack Theorem so that the distribution of RVs arriving at the terminal defense of target i (i.e., through the area defense) would look like




See Appendix F for a little further explanation.

Computationally, it would be a great benefit to have some good approximation because computing the no ipp curves is the largest computational burden. The calculation of one of the curves for a large problem was taking ≥ 5 hours of CPU time.

Appendix F contains a thorough analysis of the structure of optimal attacks, which leads to an approximation for the expected damage in the no ipp case.

## V. LARGE SCALE PROBLEMS WITH MANY AREAS - NATIONWIDE DEFENSE

We have 30 nonoverlapping areas (which we call citadels) with targets in each and independent area defenses in each.

The target structure of the citadels is shown on Figure 3. In a given row of the table, we read across the number of the citadel(s) with the particular target structure, the number of targets of value 200,100,20,10,5 and the total target value.

The total target distribution is

5 targets of value 200

10 targets of value 100

140 targets of value 20

800 targets of value 10

1000 targets of value 5

for a total of

1955 targets with total value 17,800.

I ran three different cases of the problem:

A. Area defense = 10% of target value

B. Area defense = 40% of target value

C. Area defense = 100% of target value.

In each case I computed the no ipp and ipp curves for each of the individual citadels and then used dynamic programming with a step size of 50 to determine the optimal attack against the nationwide target set.

| | Number of Targets of Given Value | | | | | |
|---|---|---|---|---|---|---|
| Citadel Number | 200 | 100 | 20 | 10 | 5 | Value |
| 1-6 | 0 | 0 | 2 | 30 | 18 | 430 |
| 7-12 | 0 | 0 | 5 | 20 | 50 | 550 |
| 13-18 | 0 | 0 | 8 | 30 | 36 | 640 |
| 19-20 | 0 | 0 | 0 | 50 | 30 | 650 |
| 21-23 | 0 | 1 | 7 | 20 | 20 | 540 |
| 24-25 | 0 | 2 | 3 | 40 | 28 | 800 |
| 26 | 1 | 0 | 2 | 20 | 50 | 690 |
| 27 | 1 | 0 | 3 | 15 | 40 | 610 |
| 28 | 1 | 0 | 10 | 15 | 30 | 700 |
| 29 | 1 | 1 | 4 | 15 | 50 | 780 |
| 30 | 1 | 2 | 4 | 15 | 30 | 780 |
| Totals: | 5 | 10 | 140 | 800 | 1000 | |

1955 Targets
17,800 Value

**Figure 3. DATA SET USED IN THE MULTI-CITADEL CASE**

A. Area defense = 10% of value

The results are shown on the following four pages as Figure 4 and Tables 1 and 2. There is approximately a 33% difference in the no ipp and ipp curves. Appendix D contains the curves comparing no ipp and ipp for the individual citadels.

B. Area defense = 40% of value

The results are shown in the same format as those of the 10% case, and are displayed as Figure 5 and Tables 3 and 4. There is approximately a 75%-85% difference between the no ipp and ipp curves. See Appendix E for further results.

C. Area defense = 100% of value

For this case, because of the lengthy computation time, I used the approximation derived in Appendix F for the no ipp curves. There is a 125% difference between the no ipp and ipp curves. See Appendix F for the results in this case.

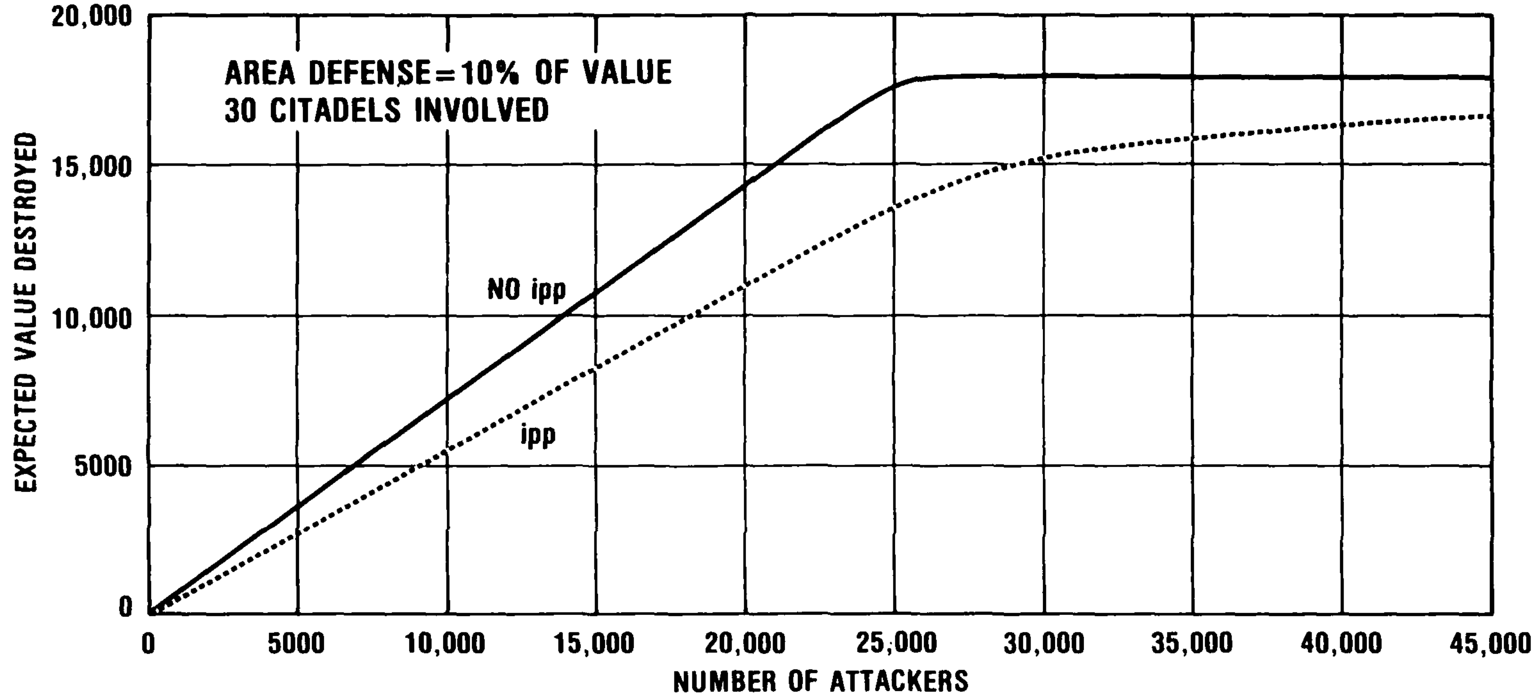



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Figure 4. Difference between ipp and no ipp for a representative nationwide target set when the area defense is equal to 10 percent of the target value.

**Table 1. Area Defense = 10% of Value**

| Value | Number of Attackers: no ipp | ipp | Difference | Percentage Difference |
|---|---|---|---|---|
| 4,450 (1/4) | 5,900 | 7,900 | 2,000 | 33.9% |
| 8,900 (1/2) | 12,150 | 16,200 | 4,050 | 33.3% |
| 13,350 (3/4) | 18,450 | 24,750 | 6,300 | 34.1% |
| 17,800 (all) | 27,450 | | | |

The percentage difference between the number of attackers needed with and without ipp to obtain three different damage levels when the area defense equals 10 percent of the target value.

Table 2. **TARGET SET OF 30 CITADELS**

Area Defense = 10 percent value

| Attackers | no ipp | ipp |
|---|---|---|
| 1000 | 773.67 | 630.00 |
| 2000 | 1537.42 | 1249.74 |
| 3000 | 2284.42 | 1817.14 |
| 4000 | 3031.42 | 2351.64 |
| 5000 | 3777.03 | 2888.61 |
| 6000 | 4515.46 | 3425.91 |
| 7000 | 5251.23 | 3962.14 |
| 8000 | 5976.14 | 4498.07 |
| 9000 | 6684.60 | 5034.15 |
| 10000 | 7393.58 | 5570.92 |
| 11000 | 8102.04 | 6106.84 |
| 12000 | 8810.53 | 6642.76 |
| 13000 | 9516.66 | 7178.31 |
| 14000 | 10223.45 | 7714.23 |
| 15000 | 10926.22 | 8251.00 |
| 16000 | 11633.89 | 8785.77 |
| 17000 | 12336.13 | 9322.53 |
| 18000 | 13034.26 | 9857.24 |
| 19000 | 13728.13 | 10392.85 |
| 20000 | 14420.93 | 10928.71 |
| 21000 | 15118.95 | 11463.16 |
| 22000 | 15807.29 | 11995.89 |
| 23000 | 16500.12 | 12508.39 |
| 24000 | 17131.27 | 12986.21 |
| 25000 | 17534.75 | 13472.32 |
| 26000 | 17721.97 | 13934.17 |
| 27000 | 17788.48 | 14347.01 |
| 28000 | 17800.00 | 14686.42 |
| 29000 | 17800.00 | 14968.50 |
| 30000 | 17800.00 | 15195.92 |
| 31000 | 17800.00 | 15397.85 |
| 32000 | 17800.00 | 15565.93 |
| 33000 | 17800.00 | 15714.10 |
| 34000 | 17800.00 | 15841.92 |
| 35000 | 17800.00 | 15956.75 |

Continued

TABLE 2. CONCLUDED

| Attackers | no ipp | ipp |
|---|---|---|
| 36000 | 17800.00 | 16057.79 |
| 37000 | 17800.00 | 16148.92 |
| 38000 | 17800.00 | 16230.27 |
| 39000 | 17800.00 | 16305.14 |
| 40000 | 17800.00 | 16372.81 |
| 41000 | 17800.00 | 16433.81 |
| 42000 | 17800.00 | 16490.35 |
| 43000 | 17800.00 | 16542.21 |
| 44000 | 17800.00 | 16590.32 |
| 45000 | 17800.00 | 16634.82 |
| 46000 | 17800.00 | 16676.11 |
| 47000 | 17800.00 | 16714.71 |
| 48000 | 17800.00 | 16750.64 |
| 49000 | 17800.00 | 16784.30 |
| 50000 | 17800.00 | 16815.93 |

Expected damage levels for different numbers of attackers with optimal attacks and defenses in the cases of ipp when the area defense equals 10 percent of the target value.

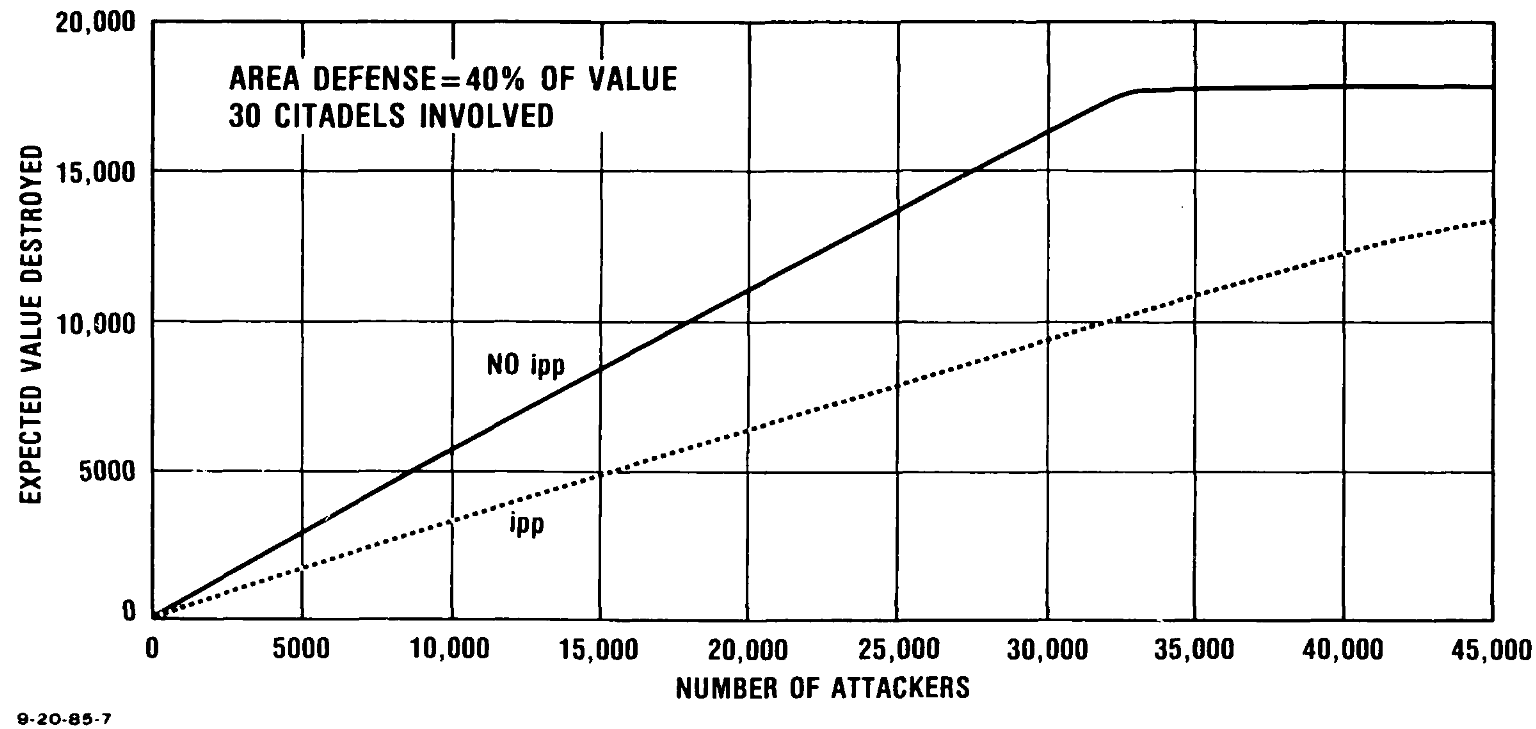



Figure 5. Difference between ipp and no ipp for a representative nationwide target set when the area defense is equal to 40 percent of the target value.

**Table 3. Area Defense = 40% of Value**

| Value | Number of Attackers no ipp | ipp | Difference | Percentage Difference |
|---|---|---|---|---|
| 4,450 (1/4) | 7,850 | 13,900 | 6,050 | 77.1% |
| 8,900 (1/2) | 16,000 | 28,600 | 12,600 | 78.8% |
| 13,350 (3/4) | 24,400 | 45,000 | 20,600 | 84.4% |
| 17,800 (all) | 35,150 | | | |

The percentage difference between the number of attackers needed with and without ipp to obtain three different damage levels when the area defense equals 40 percent of the target value.

## TABLE 4. TARGET SET OF 30 CITADELS

Area Defense = 40 percent of value

| Attackers | no ipp | ipp |
|---|---|---|
| 1000 | 571.06 | 400.00 |
| 2000 | 1141.96 | 800.00 |
| 3000 | 1708.47 | 1100.25 |
| 4000 | 2277.82 | 1438.75 |
| 5000 | 2852.13 | 1740.99 |
| 6000 | 3414.18 | 2046.36 |
| 7000 | 3977.21 | 2349.46 |
| 8000 | 4532.07 | 2652.86 |
| 9000 | 5087.93 | 2956.15 |
| 10000 | 5646.87 | 3259.54 |
| 11000 | 6188.95 | 3562.59 |
| 12000 | 6744.81 | 3865.99 |
| 13000 | 7277.76 | 4169.06 |
| 14000 | 7817.72 | 4472.23 |
| 15000 | 8352.77 | 4775.65 |
| 16000 | 8885.43 | 5078.92 |
| 17000 | 9424.26 | 5382.19 |
| 18000 | 9963.09 | 5685.22 |
| 19000 | 10491.94 | 5988.34 |
| 20000 | 11026.02 | 6291.61 |
| 21000 | 11564.85 | 6594.88 |
| 22000 | 12091.34 | 6897.80 |
| 23000 | 12623.87 | 7201.03 |
| 24000 | 13154.54 | 7504.30 |
| 25000 | 13682.89 | 7807.57 |
| 26000 | 14205.98 | 8110.49 |
| 27000 | 14729.07 | 8413.49 |
| 28000 | 15252.16 | 8716.54 |
| 29000 | 15775.25 | 9019.54 |
| 30000 | 16298.34 | 9322.54 |
| 31000 | 16821.43 | 9625.44 |
| 32000 | 17251.72 | 9928.51 |
| 33000 | 17251.72 | 1023.51 |
| 34000 | 17692.32 | 10529.98 |
| 35000 | 17791.93 | 10830.37 |

Table 4. CONCLUDED

| Attackers | no ipp | ipp |
|---|---|---|
| 36000 | 17800.00 | 11127.66 |
| 37000 | 17800.00 | 11418.10 |
| 38000 | 17800.00 | 11705.17 |
| 39000 | 17800.00 | 12002.46 |
| 40000 | 17800.00 | 12251.41 |
| 41000 | 17800.00 | 12519.97 |
| 42000 | 17800.00 | 12759.36 |
| 43000 | 17800.00 | 12973.59 |
| 44000 | 17800.00 | 13169.69 |
| 45000 | 17800.00 | 13348.16 |
| 46000 | 17800.00 | 13511.36 |
| 47000 | 17800.00 | 13662.50 |
| 48000 | 17800.00 | 13801.10 |
| 49000 | 17800.00 | 13931.17 |
| 50000 | 17800.00 | 14051.49 |

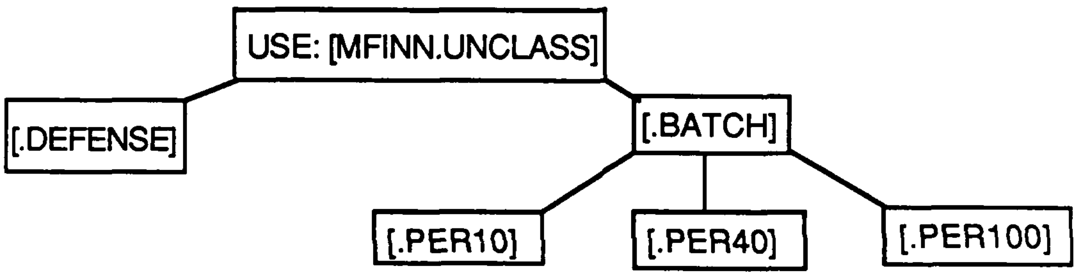
Expected damage level for different numbers of attackers with optimal attacks and defenses in the cases of ipp and no ipp when the area defense equals 40 percent of the target value.

## VI. COMPUTER PROGRAM DOCUMENTATION

All of the calculations were done on the VAX 785 system at IDA with programs written in FORTRAN.

The no ipp programs were run in a batch mode. Since I didn't (and still don't) know how to read data from a file during the execution of a FORTRAN program, each different batch job consists of the same basic program that does the work, with different "front ends" contouring the data used in that particular job. The .for program is explained in Appendix A. The other files are self explanatory.

In case other people would like to access my files, here is a diagram of my directory structures:




In the subdirectory [.DEFENSE], the following files may be found:

DYNAM10.COM - compiles, links and runs DYNAM10.FOR

DYNAM10.FOR - uses dynamic program to integrate all 30 citadels

THE10ANSWER.DAT - output from DYNAM10.FOR attack step size is 50

THESHORT10ANSWER.DAT - shortened form of the above file - step size = 1000

DYNAM40.COM
DYNAM40.FOR
THE40ANSWER.DAT
THESHORT40ANSWER.DAT

same as above except now
area defense = 40% of value instead of 10% of value

In the subdirectory [.BATCH.PER10], we find the files

RUN10NUMB%%%.*

where %%% is 001, 002, 003, ..., 011 and * is .com, .dat, .for, .log.

The numbers %%% refer to the data sets from the list of citadel structures (there are 11 citadel structures and 11 different file types). Also, area defense = 10 percent of value for each citadel.

Similarly, the subdirectories [.BATCH.PER40] and [.BATCH.PER100] contain the files

RUN4ONUMB%%%.* and RUN100NUM%%%.*

where for the first of those, area defense = 40 percent of value, and for the second, 100 percent of value.

# APPENDIX A

# DESCRIPTION AND IMPLEMENTATION OF SOLAND'S ALGORITHM

The following is a short description of a dynamic programming algorithm due to Prof. Soland that appeared in an IDA memo dated 18 July 1985. Because we are concerned with the case of perfect attackers and perfect defenders there are some simplifications.

There are A attackers, B area defenders, T targets with the value of target i equal to v(i), and at target i there are v(i) terminal interceptors.

For $i=1,2,\ldots,T$ and $x=0,1,\ldots,A$, let

$$\overline{W}_i(x) = \left\{ \begin{array}{l} \text{the expected target value destroyed at target} \\ \text{i if x RVs are assigned to it by the attacker.} \end{array} \right\}$$

Since at target i there are v(i) perfect terminal defenders, we see that

$$\overline{W}_i(x) = v(i) \cdot \left\{ \begin{array}{l} \text{the probability that more than v(i) RVs arrive at the} \\ \text{terminal defense of target i, given that x RVs were} \\ \text{aimed at it.} \end{array} \right\}$$

An elementary "ball and urn" argument shows that the distribution of the RVs through the area defense at each target is hypergeometric, so

$$\overline{W}_i(x) = v(i) \sum_{y=v(i)+1}^{x} \frac{\binom{A-B}{y}\binom{B}{x-y}}{\binom{A}{x}}$$

To find the optimal attack we first calculate $\overline{W}_i(x)$ for $i=1,2,\ldots,T$ and $x=0,1,\ldots,A$, and then apply a dynamic programming algorithm. In particular, let

$$s(u,r) = \max_{\substack{\sum_{i=1}^{u} z_i = r \\ z_i \text{ a non-negative integer}}} \sum_{i=1}^{u} \overline{W}_i(z_i)$$

for $u=1,2,\ldots,T$ and $R=0,1,\ldots,A$. We can calculate these s(u,r)'s recursively by letting

$$s(1,r) = \overline{W}_1(r)$$

for $r=0,1,\ldots,A$, and then

$$s(u,r) = \max_{z=0,1,\ldots,r} \left\{ \overline{W}_u(z) + s(u-1,r-z) \right\}$$

for $u=2,3,\ldots,T$ and $r=0,1,\ldots,A$.

The answer we are seeking is then just $s(T,A)$.

This algorithm is implemented in the FORTRAN program reproduced as Figure A-1. It is the basic program used to calculate all of the curves in the later appendices when the area defense does not have impact point prediction .

```
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN1ONUM3001.FOR;1
C
C
C                 DECLARING  THE VARIABLES
                          REAL WBAR(0:100,0:5000)
                          REAL SUM(0:100,0:5000)
                          REAL V(0:100)
                          REAL S(0:5000)
                          REAL I,J,L,X,Y,INIT,B,N,T
                          REAL A,C,D,K,PKAREA,MOON,Z
                          REAL LBIN,ANSW


        OPEN (UNIT=10, FILE='RUN1ONUM3001.DAT', STATUS='NEW')

        INIT=50
        B=43
        N=5000
        STEP=50
        T=50
        DO 2 I=1,2
                V(I)=20
2       CONTINUE
        DO 3 I=3,32
                V(I)=10
3       CONTINUE
        DO 4 I=33,50
                V(I)=5
4       CONTINUE
                MAXV=V(1)
                V(0)=V(1)+B
        WRITE (10,*)'THERE ARE',B,'AREA INTERCEPTORS WITH PK =',PKAREA
        WRITE (10,*)'THE TARGET VALUES ARE'
        WRITE (10,*) (V(I), I=1,T)
        WRITE (10,*) ' '
                DO 1 I=1,T
                        VALUE=VALUE+V(I)
1               CONTINUE

        DO 1000 A=INIT,N,STEP
                ANSW=0

C       THE CALCULATION OF WBAR(I,J) FOR PERFECT AREA INTERCEPTORS
C       (THERE ARE SHORTCUTS IN THIS CASE AS COMPARED TO THE
C        GENERAL CASE.)
                DO 42 X=0,V(0)
                 DO 41 J=0,A
                  SUM(X,J)=0.0
41               CONTINUE
42              CONTINUE
                DO 90 I=1,T
                        X=V(I)
                        Y=V(I-1)
                        IF(Y-X.LE.10**(-3))THEN
                                GOTO 90
                        ENDIF
                        DO 60 J=X+1,X+B
                            DO 55 Z=X+1,MIN(J,Y)
                             IF(A-B-.5.LT.Z)GOTO 55
                             IF(B-.5.LT.J-Z)GOTO 55
```

```
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUM8001.FOR;1
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUM8001.FOR;1
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUM8001.FOR;1
                                  IF(A-.5.LT.J)GOTO 55
        SUM(X,J)=SUM(X,J)+EXP(LBIN(A-B,Z)+LBIN(B,J-Z)-LBIN(A,J))
55                               CONTINUE
                            SUM(X,J)=SUM(X,J)+SUM(Y,J)
                            IF(SUM(X,J).GE.0.999)GOTO 80
60                          CONTINUE
                            DO 70 J=X+1+B,A
                                    SUM(X,J)=1
70                          CONTINUE
                            GOTO 90
80                          DO 85 Q=J,A
                             SUM(X,Q)=1
85                          CONTINUE
90                  CONTINUE

                    DO 92 I=1,T
                            DO 91 J=0,A
                            WBAR(I,J)=V(I)*SUM(V(I),J)
91                          CONTINUE
92                  CONTINUE



C       DYNAMIC PROGRAMMING ALGORITHM TO FIND THE OPTIMAL ATTACK
                            DO 120 R=0,A+.5
                                    S(R)=WBAR(1,R)
120                         CONTINUE

                            DO 150 U=2,T-1+.5
                                    DO 140 R=0,A+.5
                                    DO 130 Z=0,A-R+.5
                                    S(A-R)=MAX(S(A-R),WBAR(U,Z)+S(A-R-Z))
130                                 CONTINUE
140                                 CONTINUE
150                         CONTINUE
                            DO 160 Z=0,A+.5
                                    ANSW=MAX(ANSW,WBAR(T,Z)+S(A-Z))
160                         CONTINUE


C       WRITNG THE OUTPUT--EXPECTED DAMAGE BY AN OPTIMAL ATTACK

        WRITE (UNIT=10, FMT=2000) A, ANSW,(SQRT(A)-SQRT(B+1))**2
2000    FORMAT(F12.0, 2 F12.2)
                IF(ANSW.GE.VALUE*.98)GOTO 9999
1000        CONTINUE

9999            END


C       THIS FUNCTION CALCULATES THE LOGARITHM OF BINOMIAL
C       COEFFICIENTS BY USING STIRLING'S APPROXIMATION
C       TO FACTORIALS

        REAL FUNCTION LBIN(X,Y)
        IF(X.LT.0.5)THEN
```

```
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUMB001.FOR;1
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUMB001.FOR;1
_DRA2:[MFINN.UNCLASS.BATCH.PER10]RUN10NUMB001.FOR;1
                LBIN=0.0
         ELSE IF(Y.LT.0.5)THEN
                LBIN=0.0
         ELSE IF(X-Y.LT.0.5)THEN
                LBIN=0.0
         ELSE
          LBIN=LOG(SQRT(2*3.1415926*X))+X*LOG(X)-X+(1/(12*X))
          LBIN=LBIN-(LOG(SQRT(2*3.1415926*Y))+Y*LOG(Y)-Y+(1/(12*Y)))
          LBIN=LBIN-(LOG(SQRT(2*3.1415926*(X-Y))))
          LBIN=LBIN-((X-Y)*LOG(X-Y)-(X-Y)+(1/(12*(X-Y))))
         ENDIF
         RETURN

                END
```

The following presentations, Table A-1, are from the FORTRAN program in the case where the target values are 10,8,5,2,1. This particular case is the one discussed in Section II in the body of the paper.

**TABLE A-1**

Expected Value Destroyed by an optimal attack when the area defense is 5 interceptors without ipp.

Area Defense=5

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 16 | 10.00 |
| 17 | 10.00 |
| 18 | 10.51 |
| 19 | 10.85 |
| 20 | 11.50 |
| 21 | 11.87 |
| 22 | 12.27 |
| 23 | 13.07 |
| 24 | 14.07 |
| 25 | 14.80 |
| 26 | 15.02 |
| 27 | 16.81 |
| 28 | 17.59 |
| 29 | 17.91 |
| 30 | 18.41 |
| 31 | 19.12 |
| 32 | 19.59 |
| 33 | 20.33 |
| 34 | 21.51 |
| 35 | 22.27 |
| 36 | 22.60 |
| 37 | 23.18 |
| 38 | 23.85 |
| 39 | 24.36 |
| 40 | 24.79 |
| 41 | 25.22 |
| 42 | 25.46 |
| 43 | 25.66 |
| 44 | 25.80 |
| 45 | 25.91 |

Continued

## TABLE A-1

Expected Value Destroyed by an optimal attack when the area defense is 15 interceptors without ipp.

Area Defense = 15

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 26 | 10.00 |
| 27 | 10.00 |
| 28 | 10.21 |
| 29 | 10.47 |
| 30 | 10.79 |
| 31 | 11.30 |
| 32 | 11.65 |
| 33 | 12.03 |
| 34 | 13.08 |
| 35 | 13.91 |
| 36 | 14.46 |
| 37 | 15.14 |
| 38 | 16.38 |
| 39 | 17.11 |
| 40 | 17.57 |
| 41 | 17.84 |
| 42 | 18.44 |
| 43 | 18.98 |
| 44 | 19.61 |
| 45 | 20.62 |
| 46 | 21.35 |
| 47 | 21.93 |
| 48 | 22.31 |
| 49 | 22.81 |
| 50 | 23.38 |
| 51 | 23.87 |
| 52 | 24.17 |
| 53 | 24.60 |
| 54 | 24.99 |
| 55 | 25.23 |
| 56 | 25.42 |
| 57 | 25.58 |
| 58 | 25.70 |
| 59 | 25.80 |
| 60 | 25.86 |

Continued

## TABLE A-1

Expected Value Destroyed by an optimal attack when the area defense is 25 interceptors without ipp.

Area Defense = 25

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 36 | 10.00 |
| 37 | 10.00 |
| 38 | 10.11 |
| 39 | 10.33 |
| 40 | 10.59 |
| 41 | 11.09 |
| 42 | 11.47 |
| 43 | 11.71 |
| 44 | 12.69 |
| 45 | 13.55 |
| 46 | 14.13 |
| 47 | 14.63 |
| 48 | 15.77 |
| 49 | 16.65 |
| 50 | 17.21 |
| 51 | 17.56 |
| 52 | 17.91 |
| 53 | 18.49 |
| 54 | 18.93 |
| 55 | 19.83 |
| 56 | 20.64 |
| 57 | 21.32 |
| 58 | 21.80 |
| 59 | 22.16 |
| 60 | 22.68 |
| 61 | 23.21 |
| 62 | 23.62 |

Continued

TABLE A-1

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 63 | 23.95 |
| 64 | 24.39 |
| 65 | 24.70 |
| 66 | 24.98 |
| 67 | 25.20 |
| 68 | 25.39 |
| 69 | 25.53 |
| 70 | 25.65 |
| 71 | 25.74 |
| 72 | 25.81 |
| 73 | 25.85 |
| 74 | 25.89 |
| 75 | 25.92 |

Continued

**TABLE A-1**

Expected Value Destroyed by an optimal attack when the area defense is 35 interceptors without ipp.

Area Defense = 35

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 46 | 10.00 |
| 47 | 10.00 |
| 48 | 10.07 |
| 49 | 10.27 |
| 50 | 10.52 |
| 51 | 10.98 |
| 52 | 11.37 |
| 53 | 11.63 |
| 54 | 12.49 |
| 55 | 13.34 |
| 56 | 13.95 |
| 57 | 14.37 |
| 58 | 15.50 |
| 59 | 16.35 |
| 60 | 16.98 |
| 61 | 17.39 |
| 62 | 17.65 |
| 63 | 18.19 |
| 64 | 18.67 |
| 65 | 19.39 |
| 66 | 20.23 |
| 67 | 20.89 |
| 68 | 21.44 |
| 69. | 21.88 |
| 70 | 22.23 |
| 71 | 22.78 |

Continued

TABLE A-1

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 72 | 23.25 |
| 73 | 23.63 |
| 74 | 23.98 |
| 75 | 24.35 |
| 76 | 24.65 |
| 77 | 24.91 |
| 78 | 25.14 |
| 79 | 25.33 |
| 80 | 25.46 |
| 81 | 25.58 |
| 82 | 25.67 |
| 83 | 25.75 |
| 84 | 25.81 |
| 85 | 25.85 |
| 86 | 25.89 |
| 87 | 25.92 |
| 88 | 25.94 |
| 89 | 25.95 |

Continued

## TABLE A-1 Concluded

Expected Value Destroyed by an optimal attack when the area defense is 45 interceptors without ipp.

Area Defense = 45

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 56 | 10.00 |
| 57 | 10.00 |
| 58 | 10.05 |
| 59 | 10.25 |
| 60 | 10.48 |
| 61 | 10.92 |
| 62 | 11.30 |
| 63 | 11.57 |
| 64 | 12.36 |
| 65 | 13.19 |
| 66 | 13.82 |
| 67 | 14.26 |
| 68 | 15.29 |
| 69 | 16.18 |
| 70 | 16.82 |
| 71 | 17.26 |
| 72 | 17.56 |
| 73 | 17.99 |
| 74 | 18.49 |
| 75 | 19.09 |
| 76 | 19.93 |
| 77 | 20.64 |
| 78 | 21.21 |
| 79 | 21.66 |
| 80 | 22.02 |

**TABLE A-1 Concluded**

| Number of Attackers | Expected Value Destroyed |
|---|---|
| 81 | 22.50 |
| 82 | 22.97 |
| 83 | 23.39 |
| 84 | 23.72 |
| 85 | 24.08 |
| 86 | 24.41 |
| 87 | 24.69 |
| 88 | 24.93 |
| 89 | 25.15 |
| 90 | 25.31 |
| 91 | 25.41 |
| 92 | 25.57 |
| 93 | 25.66 |
| 94 | 25.73 |
| 95 | 25.79 |
| 96 | 25.84 |
| 97 | 25.87 |
| 98 | 25.90 |
| 99 | 25.93 |
| 100 | 25.94 |

# APPENDIX B

# THE SHOOT-TO-KILL STRATEGY

I have considered the following situations as in Section II:

* target values 10, 8, 5, 2, 1
* terminal defense of 10, 8, 5, 2, 1
* area defense of 5 with impact point prediction
* attack level variable
* perfect interceptors and perfect weapons.

The problem is to determine the optimal attack, i.e., the one that has the highest expected damage. R.M. Soland has written an algorithm to solve this problem. There is however a simple procedure to produce a non-trivial lower bound for the expected damage.

The procedure relies upon the "shoot-to-kill" as "guaranteed damage" strategy. With five area interceptors available to the defense, the offense can guarantee damaging the first target with 10 + 5 + 1 = 16 attackers directed at it. Similarly, the second target will be damaged by 14 attackers, the third by 11, the fourth by 8, and the fifth by 7. The "shoot-to-kill" strategy is the one where the offense picks the targets it wishes to and can destroy with its available attackers and then applies the number of attackers to those targets in order to guarantee killing them. Thus the problem reduces to simply determining the wisest choice of targets to attack.

For example, with 20 attackers the offense can damage the following combinations of targets: any of the individual targets; targets 4 and 5; targets 3 and 5; targets 3 and 4. For these possibilities, the expected damages are 1, 2, 5, 8, 10, 3, 6, 7. So, we see that the best attack is to overwhelm target 1, for an expected damage of 10.

Carrying out this procedure for a range of attackers we find the following:

## SHOOT-TO-KILL

| VALUE | B=5 | B=15 | B=25 | B=35 |
|---|---|---|---|---|
| 0 | 0-6 | 0-16 | 0-26 | 0-36 |
| 1 | 7 | 17 | 27 | 37 |
| 2 | 8-10 | 18-20 | 28-30 | 38-40 |
| 5 | 11-13 | 21-23 | 31-33 | 41-43 |
| 8 | 14-15 | 24-25 | 34-35 | 44-45 |
| 10 | 16-22 | 26-42 | 36-62 | 46-82 |
| 11 | 23 | 43 | 63 | 83 |
| 12 | 24 | 44 | 64 | 84 |
| 13 | 25-26 | 45-46 | 65-66 | 85-86 |
| 15 | 27-29 | 47-49 | 67-69 | 87-89 |
| 18 | 30-36 | 50-66 | 70-96 | 90-126 |
| 19 | 37 | 67 | 97 | 127 |
| 20 | 38-40 | 68-70 | 98-100 | 128-130 |
| 23 | 41-47 | 71-87 | 101-127 | 131-167 |
| 24 | 48 | 88 | 128 | 168 |
| 25 | 49-55 | 89-105 | 129-155 | 169-205 |
| 26 | 56+ | 106+ | 156+ | 206+ |

Number of attackers required to obtain a specific damage level using the shoot-to-kill strategy in the cases where the area defense has ipp and consists of 5,15,25,35 interceptors.

# APPENDIX C

## A MATHEMATICAL DISCUSSION OF THE KNAPSACK GAME

# A TWO LAYER ONE AREA DEFENSE WITH IMPACT POINT PREDICTION

## PERFECT ATTACKERS AND PERFECT DEFENDERS

### A. Algorithm for the Knapsack Game When There Are Many Objects of Equal Value

We shall first consider the case where all of the targets are of equal value.

There are t targets each of value V. There is a terminal defense of V perfect interceptors at each target and an area defense of B perfect interceptors. The area defense has impact point prediction, which means that the area defense can be allocated after the defense observes the attack.

The problem we will solve is to determine the minimum number of RVs the attacker needs in order to guarantee destroying r targets, where r varies between one and t.

Mathematically, the problem is to determine the least A such that

$$\underset{\substack{\sum a(i) \leq A \\ a(i)\,\varepsilon N}}{\text{MAX}} \quad \underset{\substack{\sum d(i) \leq B \\ d(i)\,\varepsilon N}}{\text{MIN}} \quad \sum f(a(i), d(i)) \geq r.$$

where

$$f(a(i),d(i)) = \begin{cases} 1 \text{ if } V+d(i) \geq a(i) \\ 0 \text{ if } V+d(i) < a(i) \end{cases}$$

and

$$N = \{0,1,2,...\}.$$

It is easy to see that for any value of A, among the set of optimal attacks will be one having for each i

$$\text{either } a(i) > V \text{ or } a(i) = 0.$$

Hence, we may restrict our attention to such attacks. Suppose then that the attacker attacks n targets, where $r \leq n \leq t$. If the attacker is to guarantee destroying r targets, then he cannot allow the defense to save (n-r)+1 targets. Thus,

(*) $$\sum_{\sigma} (a(i_{\sigma}) - v) \geq B+1$$

for any $n-r+1$ distinct subscripts $\{i_{\sigma}\}$.

If we add the $\binom{n}{n-r+1}$ different inequalities above, we find that

(**) $$\sum_{i=1}^{T} (a(i)-v) \geq \frac{n}{n-r+1} (B+1).$$

Let us renumber the targets so that

$$a(i) \leq a(j)$$

for $i < j$ .

From inequality (*) we have

$$(a(1)-v) + (a(2)-v) + \dots + (a(n-r+1)-v) \geq B+1.$$

By the pigeonhole principle

$$(a(n-r+1)-v) \geq \left\lceil \frac{B+1}{n-r+1} \right\rceil .$$

Thus,

$$a(i) - v \geq \left\lceil \frac{B+1}{n-r+1} \right\rceil \quad \text{for } i > n-r+1.$$

In order to minimize the required number of attackers, we should have

$$\sum_{i=1}^{n-r+1} (a(i)-v) = B+1$$

with

$$a(i) - v \leq \left\lceil \frac{B+1}{n-r+1} \right\rceil \quad \text{for } i=1, \dots, n-r+1$$

and

$$a(i) = \left\lceil \frac{B+1}{n-r+1} \right\rceil \quad \text{for } i = n-r+2, \dots, n.$$

We find the total number of attackers to be

$$\text{Attackers} = nV + (B+1) + (r-1)\left\lceil\frac{B+1}{n-r+1}\right\rceil$$

To get the final answer, we minimize this function over $n \in \{r, r+1, ..., t\}$.

It would be convenient to have an analytic approximation to the minimum number of attackers so as to perform analyses, draw graphs, etc. Therefore, such an approximation will be derived.

First we consider the continuous game - where a(i) and d(i) are non-negative real numbers. We can use an argument similar to the one above to find that the optimal attack is equally distributed:

$$a(i) = \frac{B+1}{n-r+1} + V \quad \text{for } i = 1,2,...,n$$

and so

$$\text{Attackers} = n\left(\frac{B+1}{n-r+1} + V\right)$$

$$= nV + (B+1) + (r-1)\left(\frac{B+1}{n-r+1}\right).$$

We now further relax the constraint of n being integral. Thus, the offense is free to choose the exact value of targets to be attacked. For V small and t large this is probably a valid assumption.

In any event, we shall use the following combination of notations:

$A_{int}$ refers to the problem where $a(i), d(i) \in N$

$A_{cont}$ refers to the continuous problem

$\overline{A}$ refers to the problem where $n \in N$.

Thus, $A_{int}$ would be the answer to the original problem, while $A_{cont}$ is the most tractable analytically.

To find $A_{cont}$, we rewrite the number of attackers as

$$\text{Attackers} = (r-1)V + (B+1) + (n-r+1)V + (r-1)\frac{B+1}{n-r+1} = f(n).$$

The first two terms are constant, so we don't need to consider them. To minimize the latter terms, we use the Arithmetic Mean-Geometric Mean inequality which says that the sum of two positive numbers whose product is a constant is minimized when the numbers are equal, which implies

$$(n^*-r+1)V = (r-1)\frac{B+1}{n^*-r+1}$$

or,

$$n^*-r+1 = \sqrt{(r-1)(B+1)/V}$$

So,

$$A_{cont} = (r-1)V + (B+1) + 2\sqrt{(r-1)(B+1)V}$$
$$= (\sqrt{(r-1)V} + \sqrt{(B+1)})^2,$$

unless of course

$$n^*-r+1 = \sqrt{(r-1)(B+1)/V} \geq t-r+1$$

in which case the minimum occurs at n=t, so

$$A_{cont} = tV + \frac{t}{t-r+1}(B+1).$$

In order to find $A_{cont}$, we know that $n^*$ is at most 1/2 away from the nearest integer. And we know that f"(u) is decreasing, so

$$\bar{A}_{cont} \leq \max\ (f(n^*+1/2), f(n^*-1/2)) = f(n^*-1/2).$$

Substituting this in, we see that

$$\bar{A}_{cont} \leq A_{cont} + (f(n^*-1/2) - A_{cont})$$

$$= A_{cont} + \frac{1/4\ V}{\sqrt{\frac{(r-1)(B+1)}{V}} - 1/2}$$

$$= A_{cont} + \frac{V}{4(n^*-r+1/2)}.$$

If $n^* \geq t$, then

$$\bar{A}_{cont} = A_{cont}.$$

Finally, to get a bound on $A_{int}$, we see that for all n,

$$(r-1)\left(\frac{B+1}{n-r+1}\right) \leq (r-1)\left\lceil\frac{B+1}{n-r+1}\right\rceil \leq (r-1)\left(\frac{B+1}{n-r+1}\right) + (r-1).$$

Therefore, we have

$$\overline{A}_{cont} \leq \overline{A}_{cont} \leq \overline{A}_{cont} + (r-1).$$

Combining all three inequalities, we get

$$A_{cont} \leq \overline{A}_{cont} \leq \overline{A}_{int} \leq \overline{A}_{cont} + (r-1) \leq A_{cont} + (r-1) + \frac{V}{4(n^*-r-1)}$$

and if $n^* \leq t$,

$$A_{cont} = \overline{A}_{cont} \leq \overline{A}_{int} \leq \overline{A}_{cont} + (r-1) = A_{int} + (r-1).$$

It is possible to derive a lower bound for $\overline{A}_{int}$ by considering $A_{int}$.

The number of attackers for the integral solution is

$$\text{Attackers} = (r-1)V + (B+1) + (n-r+1)V + (r-1)\left\lceil\frac{B+1}{n-r+1}\right\rceil$$

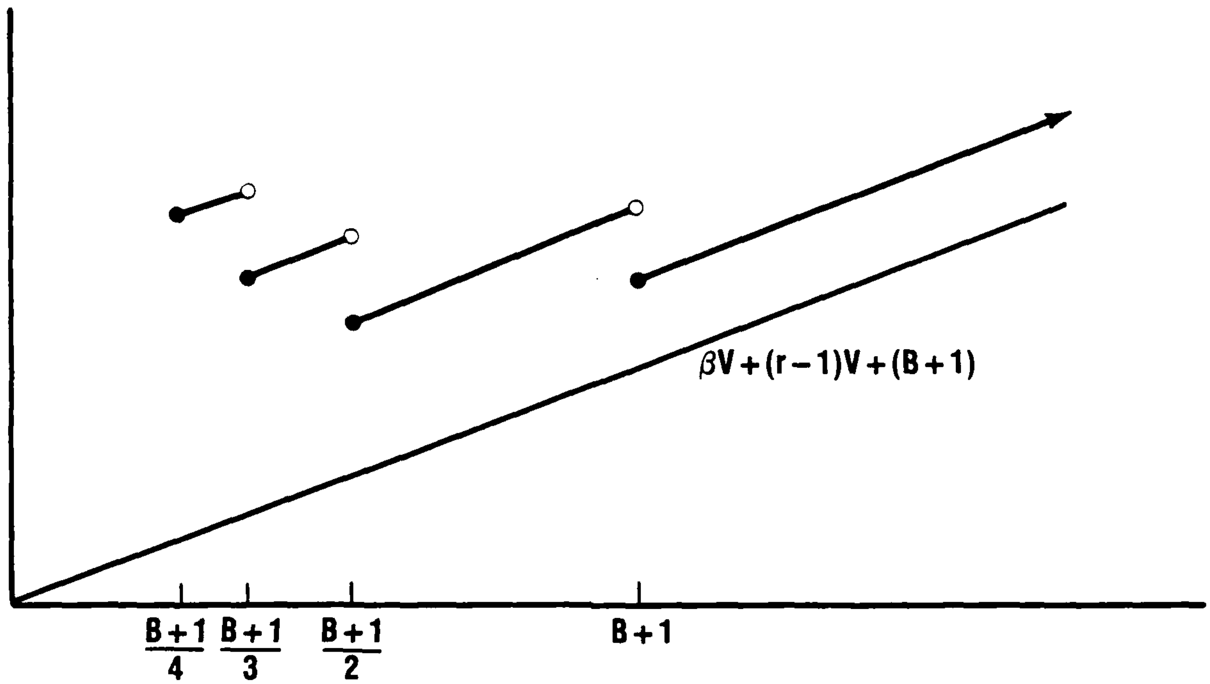
which we wish to minimize for $n \varepsilon [r,t]$.

Letting $\beta = n-r+1$, so that $\beta \varepsilon [1,t-r+1]$, the number of attackers can be rewritten as

$$\text{Attackers} = (r-1)V + (B+1) + \beta V + (r-1)\left\lceil\frac{B+1}{\beta}\right\rceil.$$

The graph of this function is a series of line segments of slope V, with end points at $\beta = \frac{B+1}{i}$ and $\beta = \frac{B+1}{i-1}$

which looks something like




10-8-85-1

The minima will obviously occur at $\frac{B+1}{i}$ where i is the largest integer such that

$$V\left(\frac{B+1}{i-1} - \frac{B+1}{i}\right) \geq (r-1)$$

i.e.,

$$i^* = \left\lfloor \frac{1 + \sqrt{1 + \frac{4V(B+1)}{r-1}}}{2} \right\rfloor$$

unless $\frac{B+1}{i^*} > t\text{-}r+1$, in which case

$$i^* = \left\lceil \frac{B+1}{t-r+1} \right\rceil .$$

In either case,

$$A_{int} = (i^*+V)\,(r-1 \; + \frac{B+1}{i^*})\ ,$$

which is another lower bound for $\overline{A}_{int}$.

## B. Algorithm for the Knapsack Game When There Are Many Objects of Unequal Value

Using the insight gained from section A, we can now outline an algorithm for the case of unequal target values.

Suppose the targets are valued v(i) with

$$v(1) \geq v(2) \geq \ldots \geq v(T).$$

One way to solve the problem is to postulate an attack level A and then enumerate all of the attack vectors

$$a = [a(1), a(2), \ldots , a(T)]$$

with

$$0 \leq a(i) \leq A$$

and

$$\sum a(i) = A.$$

For each one of these attack vectors, the defense has a knapsack problem to solve, i.e.,

$$\text{MAX} \qquad \sum z_i v_i$$

such that

$$\sum z_i f_i(a(i)) \leq B$$

$$z_i \varepsilon \{0,1\}$$

where

$$f_i(x) = \max(x\text{-}v(i),0).$$

The number of attack vectors to be examined (and hence knapsack problems to be solved) is on the order of $A^{T-1}$, which is beyond feasibility when considering attacks on large target sets.

In the case of perfect attackers and interceptors, it is simple to prove that optimal attacks are monotone in their excesses on targets attacked. In other words, if i<j (so $v(i) \geq v(j)$) and $a(i) = 0$ $a(j) = 0$, then

$$a(i) - v(i) \geq a(j) - v(j).$$

Using this we can diminish the number of attack vectors considered by roughly a factor of T!, so the number of knapsack problems to be solved is still on the order of $A^{(T-1)}$.

Sabbagh's implicit enumeration algorithm does not fare any better because the "jumps" in the implicit enumeration scheme are not sufficiently large to significantly decrease the number of attack vectors considered.

The algorithm outlined below attempts to circumvent the computational size of the above algorithms <u>in the case where there are many targets of the same value</u> by means similar to the ones employed in part A. The algorithm must solve at most

$$\prod_{i=1}^{G} (n(i) + 1)$$

linear integer programming problems, where n(i) is the number of targets of value v(i) and G is the number of distinct target values.

In order to make the discussion as concrete as possible, we will consider the following problem:

6 targets of value 4

4 targets of value 7

Required damage level = 25

Area defense = B

We wish to determine the minimum number of attackers needed to obtain a value destroyed of 25.

Figure C-1 shows the set up for the algorithm. Along the x-axis is the number of targets of value 7 and along the y-axis is the number of targets of value 4. The boundary line for u=25 satisfies the equation

$$\frac{x}{25/7} + \frac{y}{25/4} = 1 \ .$$

Points to its left represents target combinations that sum to less than 25 value and points to its right more than 25. The 19 integer points on the line or to its right represent feasible attack points. We shall examine the one circled, which represents the attacker attacking three targets of value 7 and 4 targets of value 4. Let $x_i(i=1,2,3)$ be the excesses of the attacks against the targets of value 7 and $y_j(j=1,...,4)$ be the excesses of the attacks against the targets of value 4.

The shaded region is the "defense wins" region, which means that if the defender can save a combination of targets to get within the region, then the postulated attack does not produce a destruction of 25. For example, if the defense can save 3 targets of value 4 under attack and 2 targets of value 7 under attack, then the offense destroyed only 1 target of value 4 and 1 of value 7, hence a total of 11, which is less than 25.

Hence, points in the region represent constraints on the $x_i$'s and $y_j$'s.

The minimal points represent the smallest set of irredundant constraints. They are defined as the minimal set of points generating the whole "defense wins" region, where the point $(\alpha,\beta)$ "generates" the rectangular set of points $(x,y)$ satisfying $(x,y) \varepsilon [0,\alpha] \times [0,\beta]$.

The uppermost minimal point represents the constraint that the defense cannot be allowed to save any two of the targets of value 7. Hence

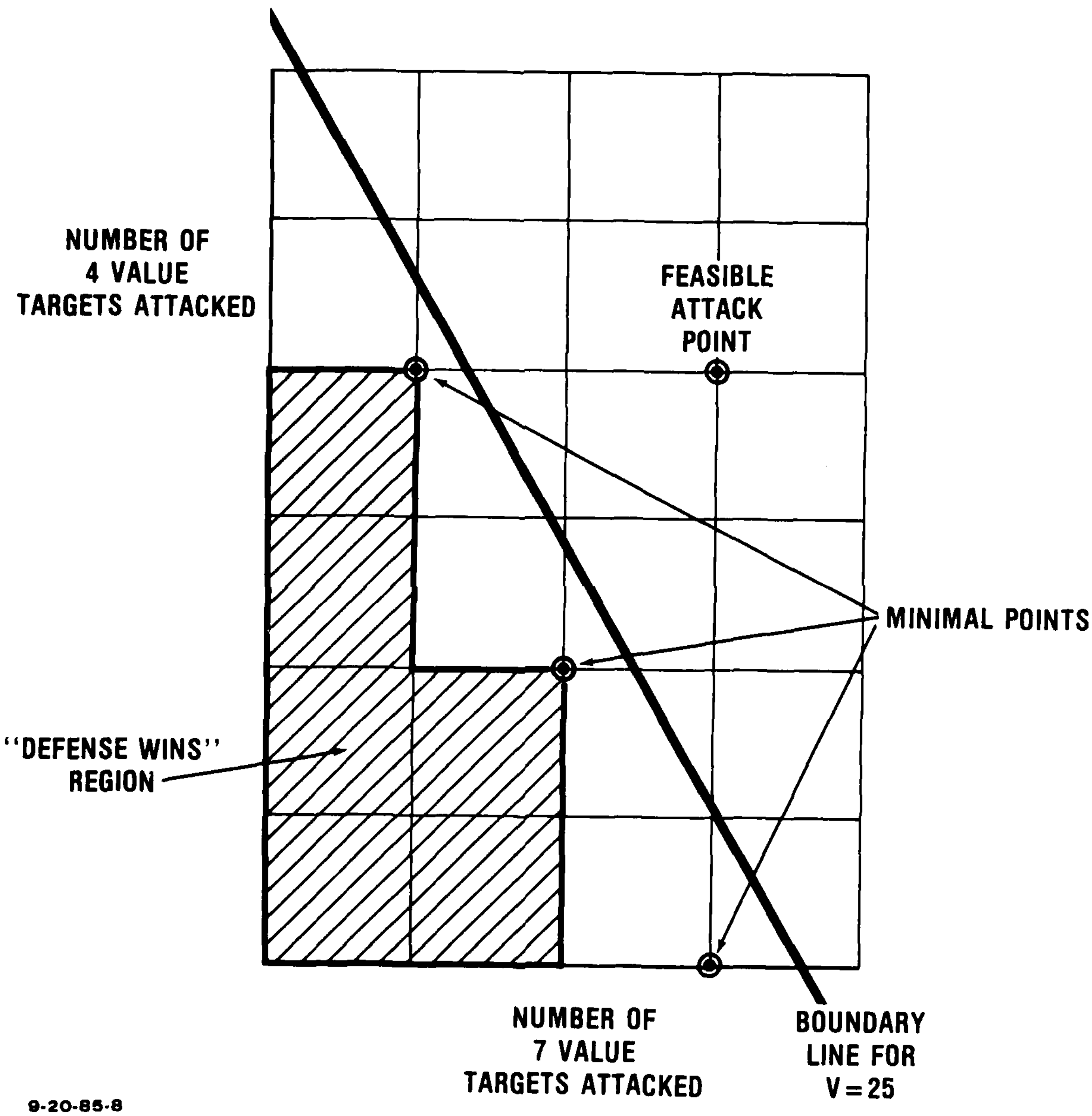



Figure C-1. Geometric Structures in the Knapsack Game Algorithm

$$x_1 + x_2 \geq (B+1)$$

$$x_1 + x_2 \geq (B+1)$$

$$x_2 + x_3 \geq (B+1).$$

The middle minimal point represents the constraint that the defense cannot be allowed to save 1 target of value 7 and 2 targets of value 4. Hence

$$x_\alpha + y_{\beta_1} + y_{\beta_2} \geq (B+1)$$

for $\alpha \, \varepsilon \{1,2,3\}$ and $\beta_1, \beta_2$ distinct elements of {1,2,3,4] .

The final set of constraints is that the defense cannot be allowed to save all 4 targets of value 4. Thus,

$$y_1 + y_2 + y_3 + y_4 \geq (B+1).$$

What we need to do now is minimize the total attack,

$$3 \times 7 + x_1 + x_2 + x_3 + 4 \times 4 + y_1 + y_2 + y_3 + y_4$$

subject to these constraints.

By summing the first constraints, dividing by 2 and adding the last constraint we find

$$x_1 + x_2 + x_3 + y_1 + y_2 + y_3 + y_4 \geq \frac{5}{2} (B+1)$$

with equality if

$$x_j = 1/2(B+1), \; y_i = 1/4(B+1).$$

All of the middle constraints are satisfied with these values of $x_i$ and $y_j$ so we have found a (potentially non integral) optimal attack. If B=3 (mod 4) then the attack is integral. If not, we can get good bounds on the number of attackers required by decreasing B by i and increasing B by 4-i where B=i (mod 4), $i \varepsilon \{1,2,3\}$.

Then we vary the feasible attack point (circled point) to one of the 18 other feasible points. At each point we solve the problem and take the minimum of all the answers to get the final result.

This algorithm generalizes immediately to more than 2 distinct target values.

## C. Remarks and Simplications of the Algorithm

1. Suppose the feasible attack point under consideration is $(\alpha,\beta)$ and we are interested in the constraint due to the point $(\alpha-a, \beta-b)$. Then we know that

$$\sum x_{i_\pi} + \sum y_{j_\sigma} \geq (B+1)$$

where

$\{i_\pi\}$ is any collection of a distinct indices in $\{1,2, \dots, \alpha\}$ and $\{j_\sigma\}$ is an distinct collection of b indices in $\{1,2, \dots, \beta\}$. If we add up all of the possible combinations, we find that

$$\binom{\beta}{b} \cdot \left\{ \binom{\alpha}{a} \cdot \frac{a}{\alpha} \sum_{i=1}^{d} x_i \right\} + \binom{\alpha}{a} \left\{ \binom{\beta}{b} \cdot \frac{b}{\beta} \sum_{j=1}^{\beta} y_j \right\} \geq \binom{\beta}{b} \binom{\alpha}{a} (B+1)$$

so that

$$\frac{a}{\alpha} \sum x_i + \frac{b}{\beta} \sum y_i \geq (B+1).$$

If we now let $a,b,\alpha,\beta$ assume continuous values, we can get an analytical lower bound. Suppose we take a and b so that

$$\frac{\alpha}{a} = \frac{\beta}{b}$$

and

$$(\alpha\text{-}a) \times 7 + (\beta\text{-}b) \times 4 = 25 - \varepsilon$$

$\varepsilon$ = small constant.

In the limit, $a = \alpha - \frac{25}{7+4\,\beta/\alpha}$ and $b = \beta - \frac{25}{7\alpha/\beta+4}$.

Then the constraint becomes

$$\left(1 - \frac{25}{7\alpha + 4\,\beta}\right) \left(\sum x_i + \sum y_j\right) \geq (B+1)$$

which implies $\sum x_i + \sum y_j \geq \dfrac{B+1}{1 - \dfrac{25}{7\alpha + 4\beta}}$.

Thus

$$\text{Att} \geq \frac{B+1}{1 - \dfrac{25}{7\alpha+4\beta}} + 7\alpha + 4\beta.$$

If we minimize this with respect to $\alpha,\beta$, we find it's a minimum when

$$7\alpha + 4\beta = 25\sqrt{25\,(B+1)}$$

which implies that

$$\text{Att} \geq \frac{B+1}{1 - \dfrac{25}{25 + \sqrt{25\,(B+1)}}} + 25 + \sqrt{25\,(B+1)}$$

$$= (\sqrt{25} + \sqrt{B+1}\,)^2,$$

unless $25 + \sqrt{25(B+1)} \geq$ TOTAL VALUE $= 4 \times 7 + 6 \times 4 = 52$,

in which case we would take

$$7\alpha + 4\beta = \text{TOTAL VAL},$$

so that

$$\text{Att} \geq \frac{B+1}{1 + \dfrac{25}{\text{TOT VAL}}} + \text{TOT VAL}$$

$$= \text{TOT VAL}\left\{1 + \frac{B+1}{\text{TOT VAL} - 25}\right\}.$$

$$= 52\left\{1 + \frac{B+1}{27}\right\}$$

2. The algorithm can be simplified somewhat if the exact integral answer is not desired but just a tight bound on it.

Suppose there are N distinct target values.

We shall keep the same terminology as in the case of two target groups, except we work in n-dismensional space where the boundary line becomes a boundary hyperplane. Suppose we are investigating the feasible attack point $(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n)$ and are concerned with the constraints due to the vector $(b_1,b_2,...,b_n)$. As above, if we add all of the possible inequalities, we find

$$\sum_{i=1}^{N}\left(1-\frac{b_i}{\alpha_i}\right)\sum x_{i,k} \geq B+1.$$

Let C be the convex hull of the lattice points in the "defense wins" region. Let

$\vec{p}=\lambda\vec{\alpha}$

where

$$\lambda = \sup\{\lambda: \lambda\vec{\alpha}\,\varepsilon\, C\} .$$

Write p as

$$\vec{p}=\sum_{j=1}^{n} w_j \cdot (\vec{b}_j)$$

where $\vec{b}_j$ = a lattice point in the defense wins region

$$0 \leq w_j \leq 1, \sum w_j = 1.$$

i.e., find p's barycentric coordinates.

Since each of the $b_j$'s is a constraint vector, we have for all j

$$\sum_{i=1}^{N} (1 -\frac{b_{j,i}}{\alpha_i}) \sum_{k} x_{i,k} \geq B+1.$$

If we multiply the jth equation by $w_j$ and add, we find

$$\sum_{i=1}^{N}\left(1-\frac{\Sigma_j w_j b_{j,i}}{\alpha_i}\right)\sum x_{i,k} \geq B+1$$

which can be rewritten

$$\sum_{i=1}^{N}\left(1-\frac{P_i}{\alpha_i}\right)\sum x_{i,k} \geq B+1 \quad .$$

Thus

$$\sum(1-\lambda)\sum x_{i,k} \geq B+1$$

and

$$\sum\sum x_{i,k} \geq \frac{B+1}{(1-\lambda)} \quad .$$

Let

$$-D+A_1z_1+..+A_nZ_n=\det\begin{vmatrix} z_1-b_{1,1} & z_2-b_{1,2} & \dots & z_n-b_{1,n} \\ a_{2,1}-b_{1,1} & b_{2,2}-b_{1,2} & \dots & b_{2,n}-b_{1,n} \\ & \vdots & & \\ b_{n,1}-a_{1,1} & b_{n,2}-a_{1,2} & \dots & b_{n,n}-b_{1,n} \end{vmatrix}=0$$

be the equation of the hyperplane through the points $b_1, \dots, b_n$.

If we let

$$\sum_K x_{i,k} = -\frac{B+1}{\dfrac{(A_1\alpha_1+..\ +A_i\hat{\alpha}_i+\dots\ +A_n\alpha_n)-D}{A_i}}$$

then all of the constraints will be satisfied, and we will have equality in the constraint for $\sum\sum x_{i,k}$, so we have obtained the minimum.

However,

$$x_{i,k} = \frac{t_i}{s_i}(B+1)$$

where $t_i, s_i \varepsilon Z$.

This is the rational solution.

The integral solution must have $x_{i,k} \varepsilon Z$. We can obtain an upper bound on the number of attackers by letting

$$\left.\begin{array}{l} x_{i,1} = \\ x_{1,2} = \\ x_{i,3} = \\ \bullet \\ \bullet \\ \bullet \\ x_{i,r} = \end{array}\right\} = \left\lfloor \frac{t_i}{s_i} (B+1) \right\rfloor$$

$$+$$

$$\left.\begin{array}{l} x_{i,r+1} = \\ \bullet \\ \bullet \\ \bullet \\ x_{i,\alpha_i} = \end{array}\right\} = \left\lceil \frac{t_i}{s_i} (B+1) \right\rceil$$

where $r = (\alpha_i - b_i) \left\lceil \frac{t_i}{s_i} (B+1) \right\rceil - \left\lceil (\alpha_i - b_i) \frac{t_i}{s_i} (B+1) \right\rceil$ .

Then the total number of attackers assigned to the ith target group is

$$\mathrm{Att}_i = \alpha_i v_i + \left\lceil (\alpha_i - b_i) \frac{t_i}{s_i} (B+1) \right\rceil + b_i \left\lceil \frac{t_i}{s_i} (B+1) \right\rceil$$

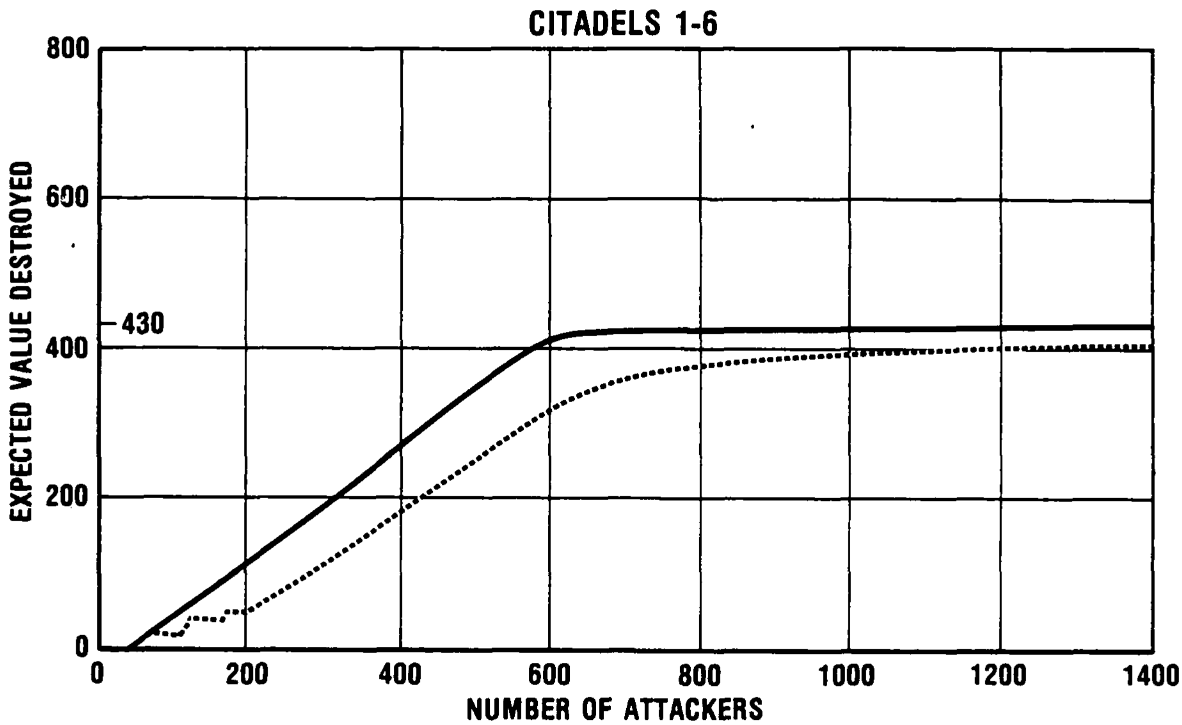
Thus we have the bound

$$\begin{aligned}
\alpha_i v_i + d_i \frac{t_i}{s_i} (B+1) \le \mathrm{Att}_i &\le \alpha_i v_i + \left\lceil (\alpha_i - b_i \frac{t_i}{s_i} (B+1) \right\rceil \\
&\quad + b_i \left\lceil \frac{t_i}{s_i} (B+1) \right\rceil \\
&\le \alpha_i v_i + (\alpha_i - b_i) \frac{t_i}{s_i} (B+1) + 1 \\
&\quad + b_i \frac{t_i}{s_i} (B+1) + b_i \\
&= \alpha_i v_i + \alpha_i \frac{t_i}{s_i} (B+1) + (b_i + 1)
\end{aligned}$$

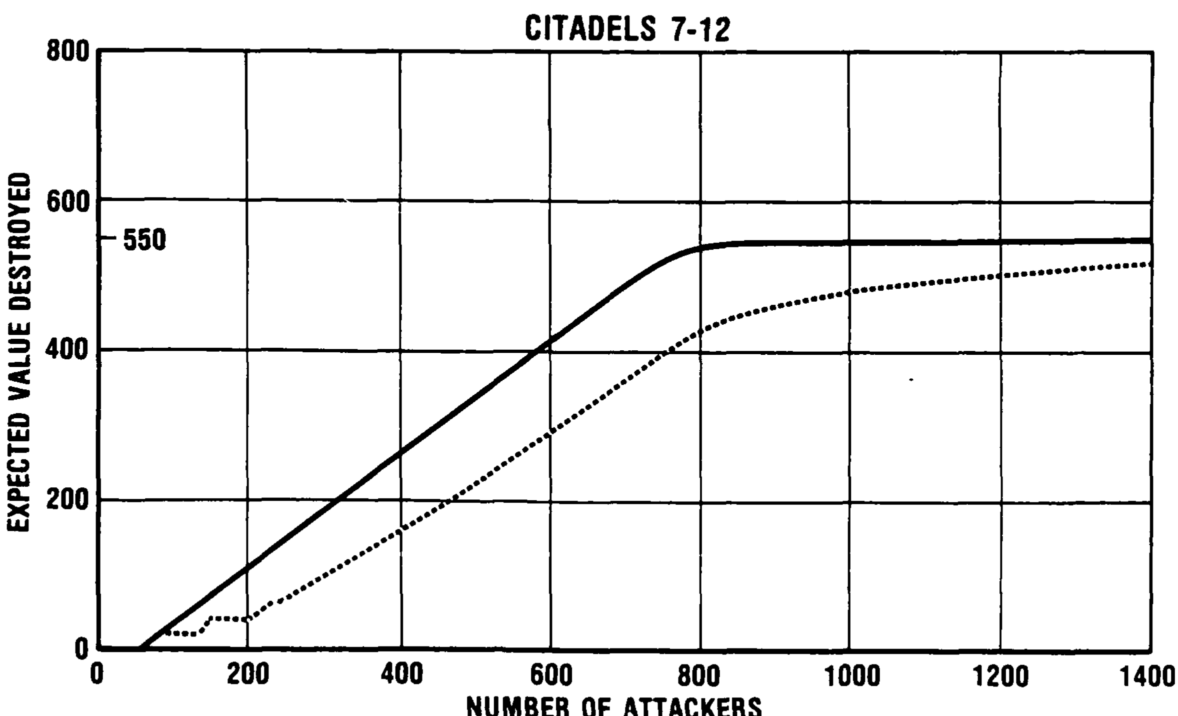
# APPENDIX D

## RESULTS WHEN AREA DEFENSE EQUALS 10 PERCENT OF TARGET VALUE

The twelve charts contained in this appendix demonstrate the difference between ipp and no ipp for each of the thirty citadels when the area defense equals ten percent of the target value. On each chart, the solid line represents no ipp and the dashed line represents ipp. The citadels covered by a specific chart are noted at the top of the chart.

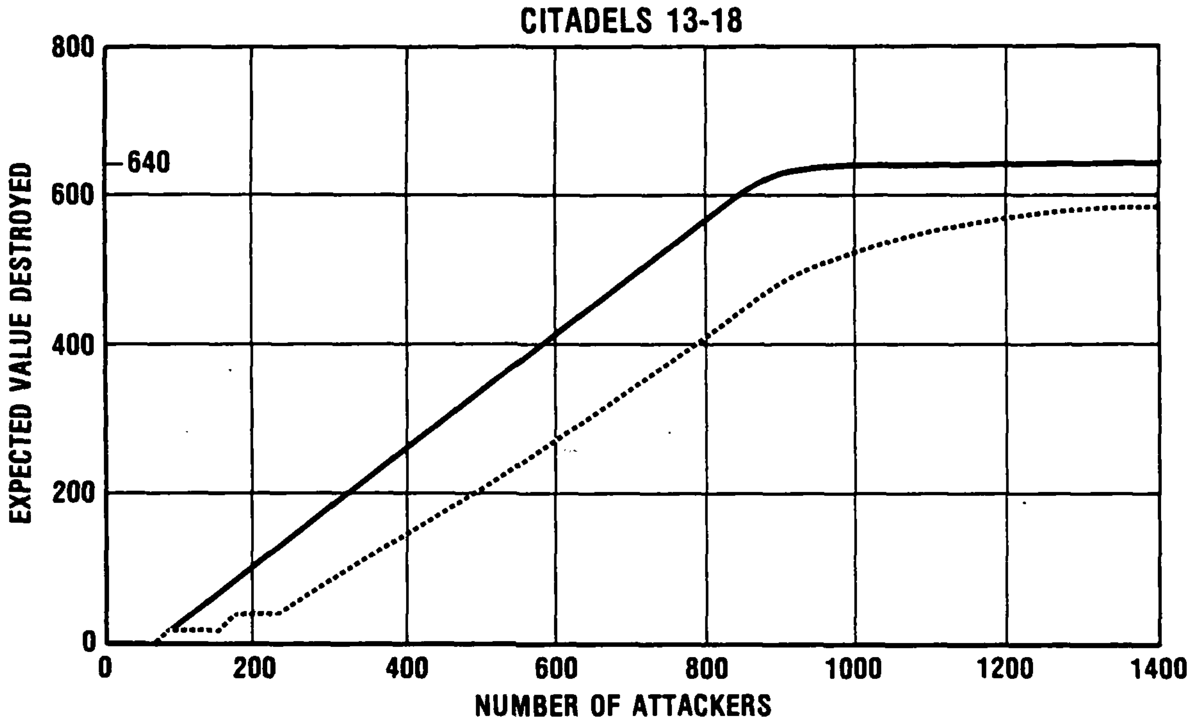


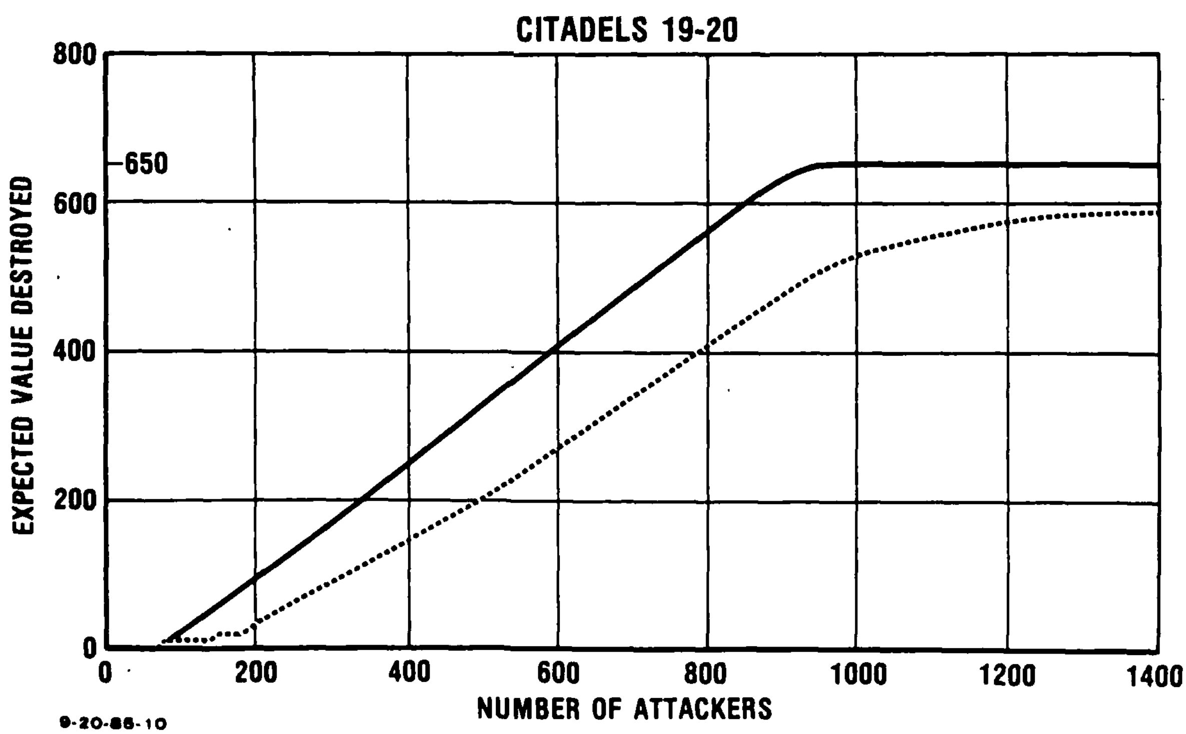
CITADELS 1-6
EXPECTED VALUE DESTROYED
800
600
430
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

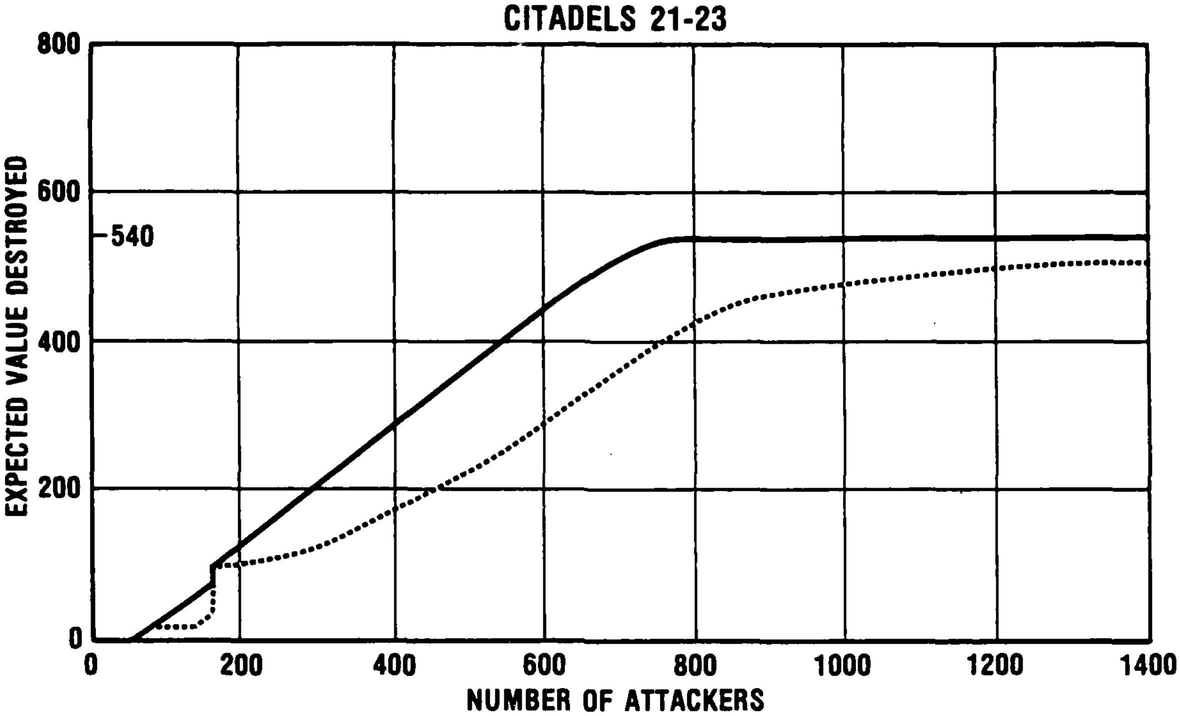
CITADELS 7-12
EXPECTED VALUE DESTROYED
800
600
550
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

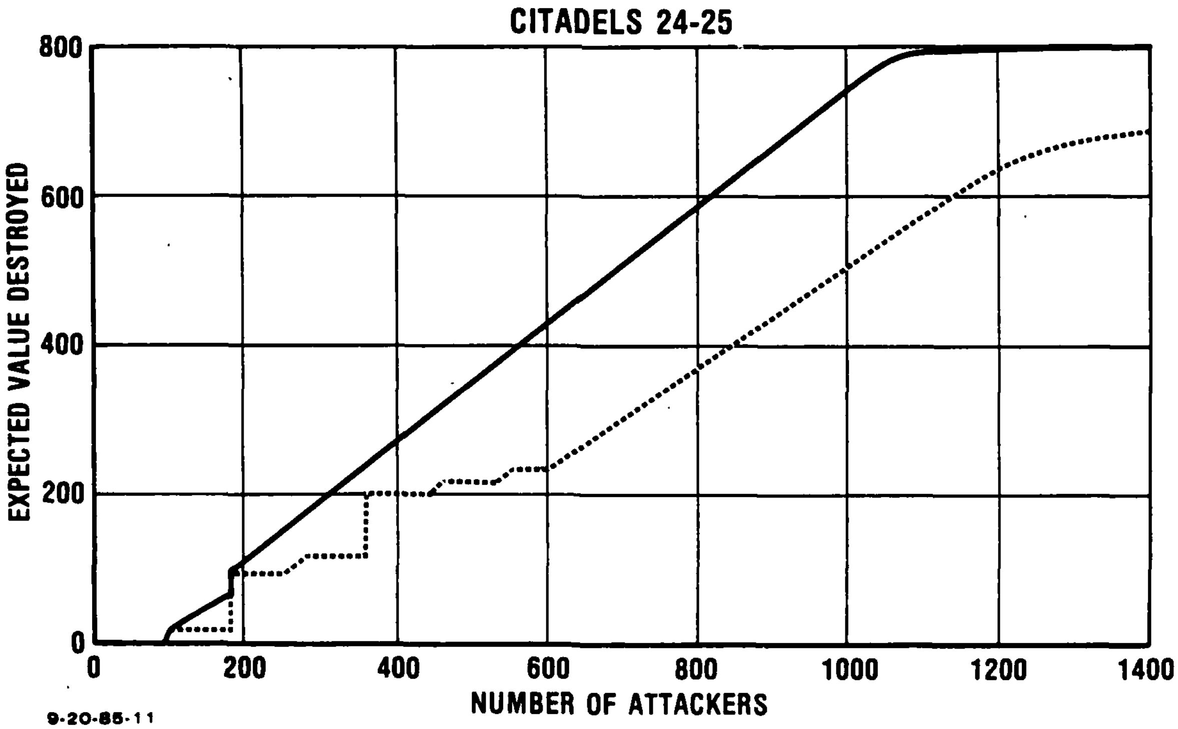

9-20-85-9

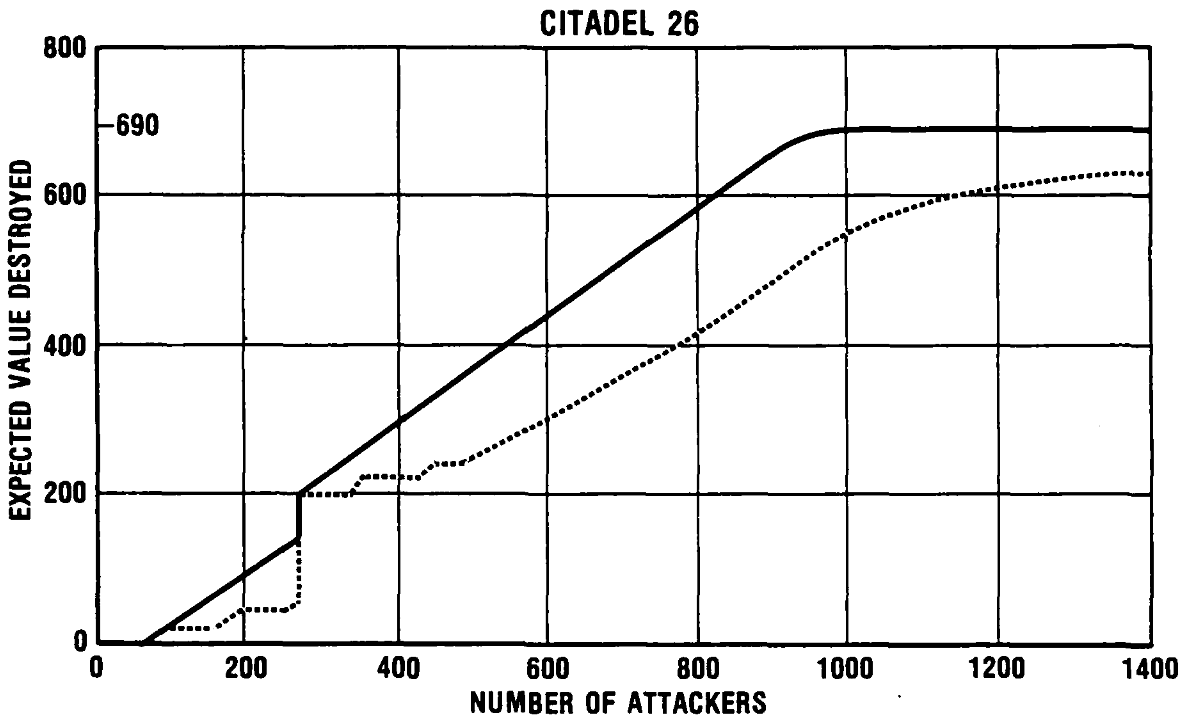

CITADELS 13-18
EXPECTED VALUE DESTROYED
800
640
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

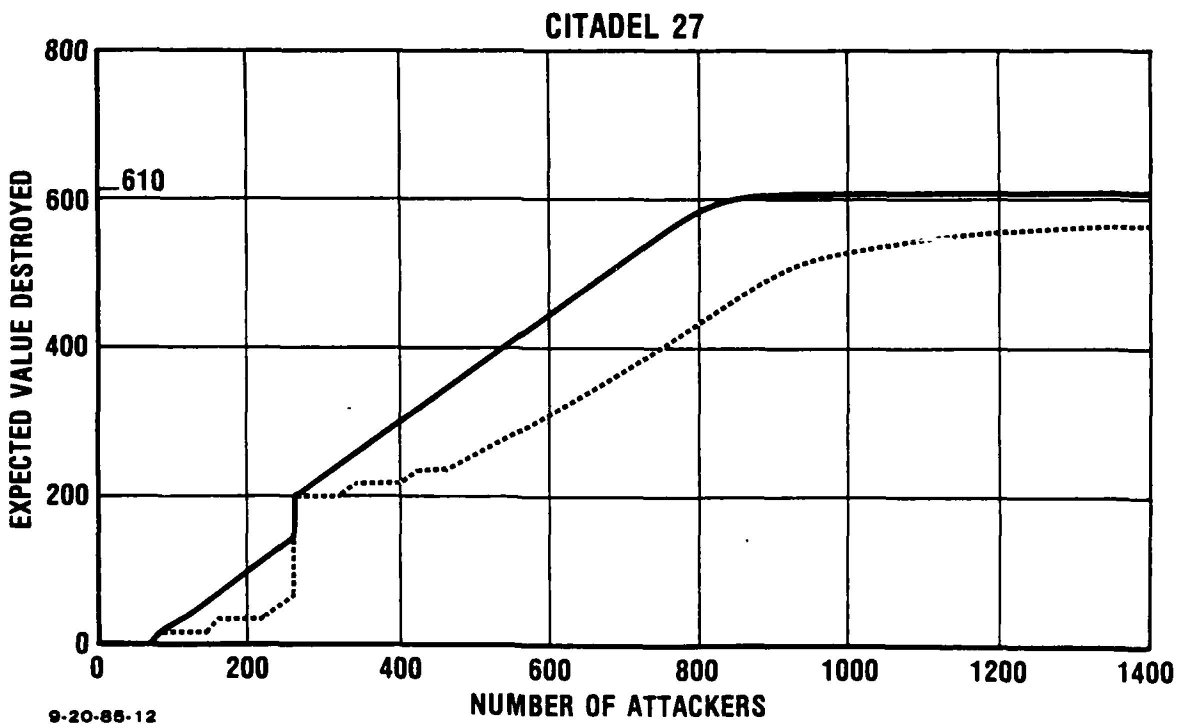

CITADELS 19-20
EXPECTED VALUE DESTROYED
800
650
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

9-20-86-10

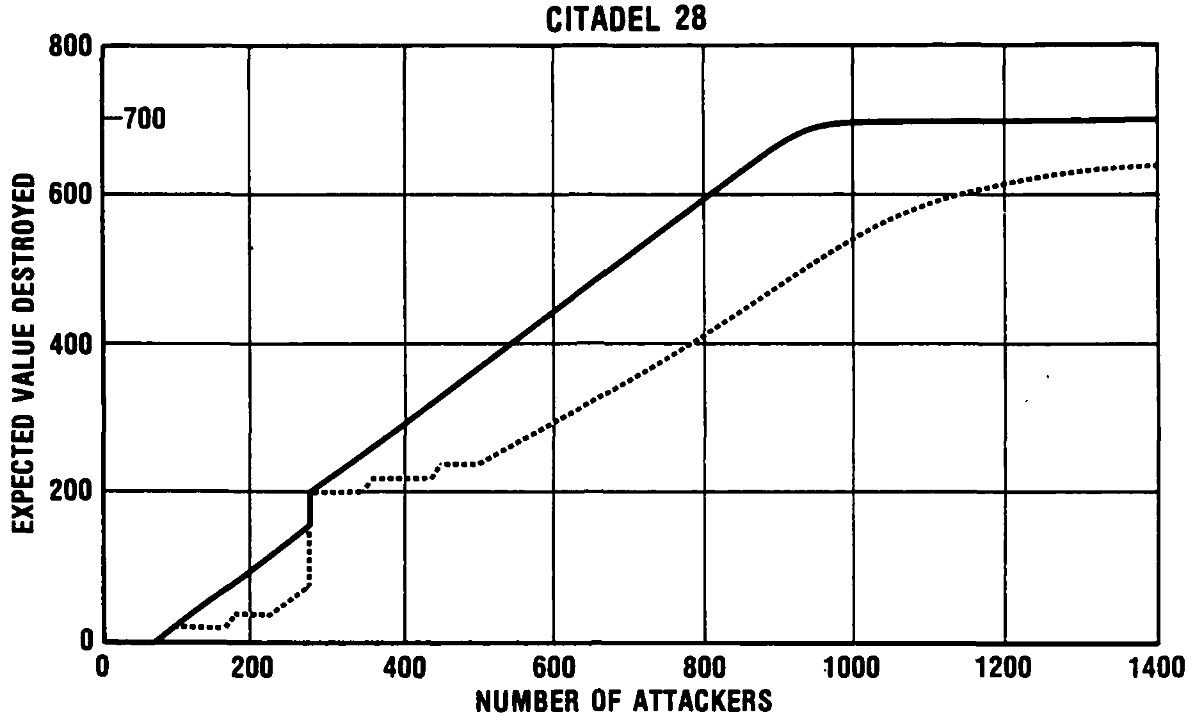

CITADELS 21-23
EXPECTED VALUE DESTROYED
800
600
540
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

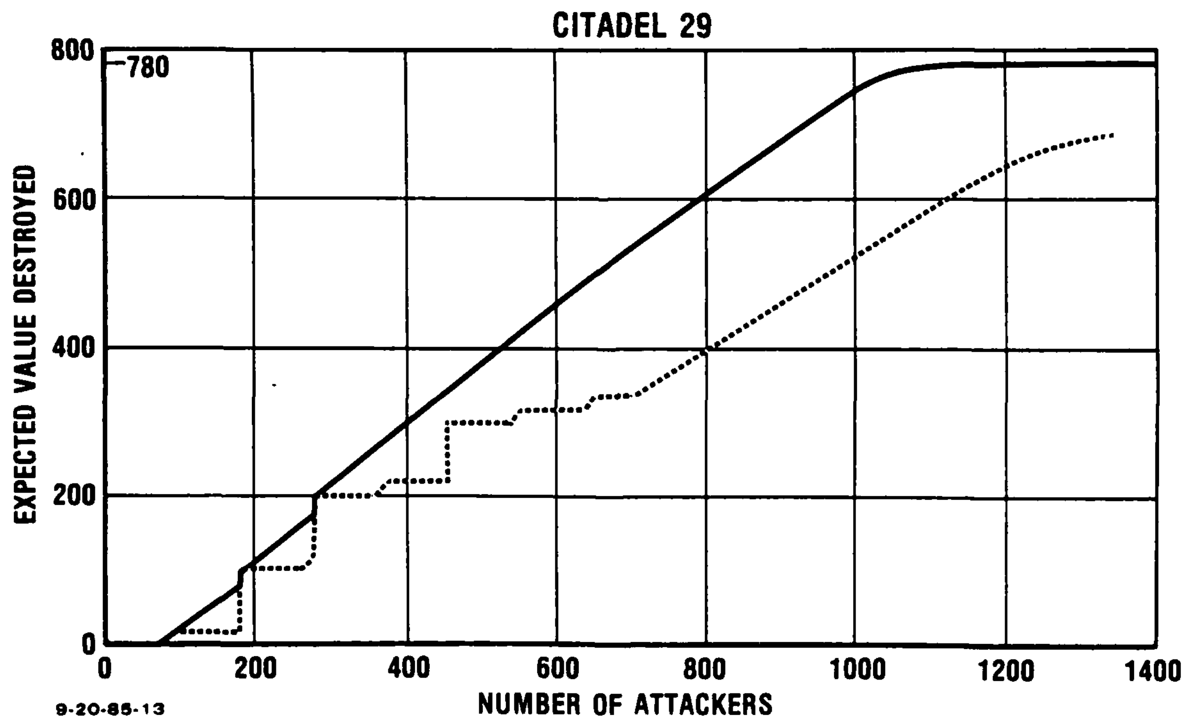

CITADELS 24-25
EXPECTED VALUE DESTROYED
800
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS
9-20-85-11


CITADEL 26
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
690
0
200
400
600
800
1000
1200
1400

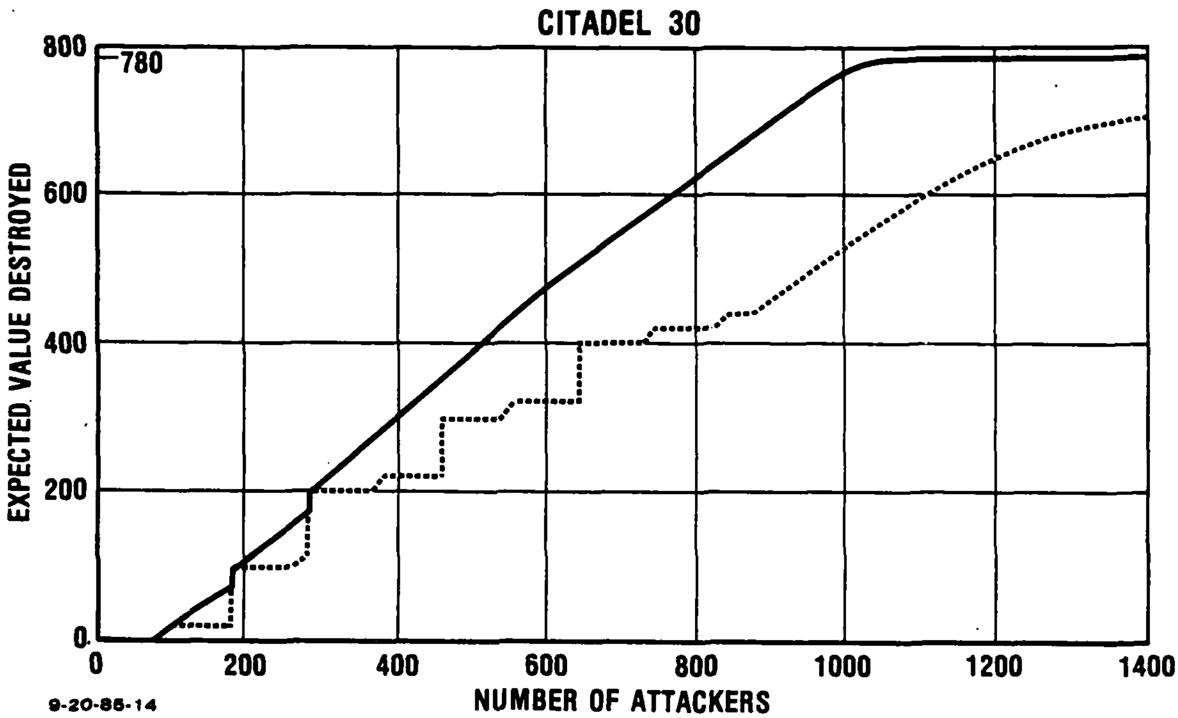

CITADEL 27
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
610
0
200
400
600
800
1000
1200
1400

9-20-85-12


CITADEL 28
EXPECTED VALUE DESTROYED
800
700
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS


CITADEL 29
EXPECTED VALUE DESTROYED
800
780
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

9-20-85-13

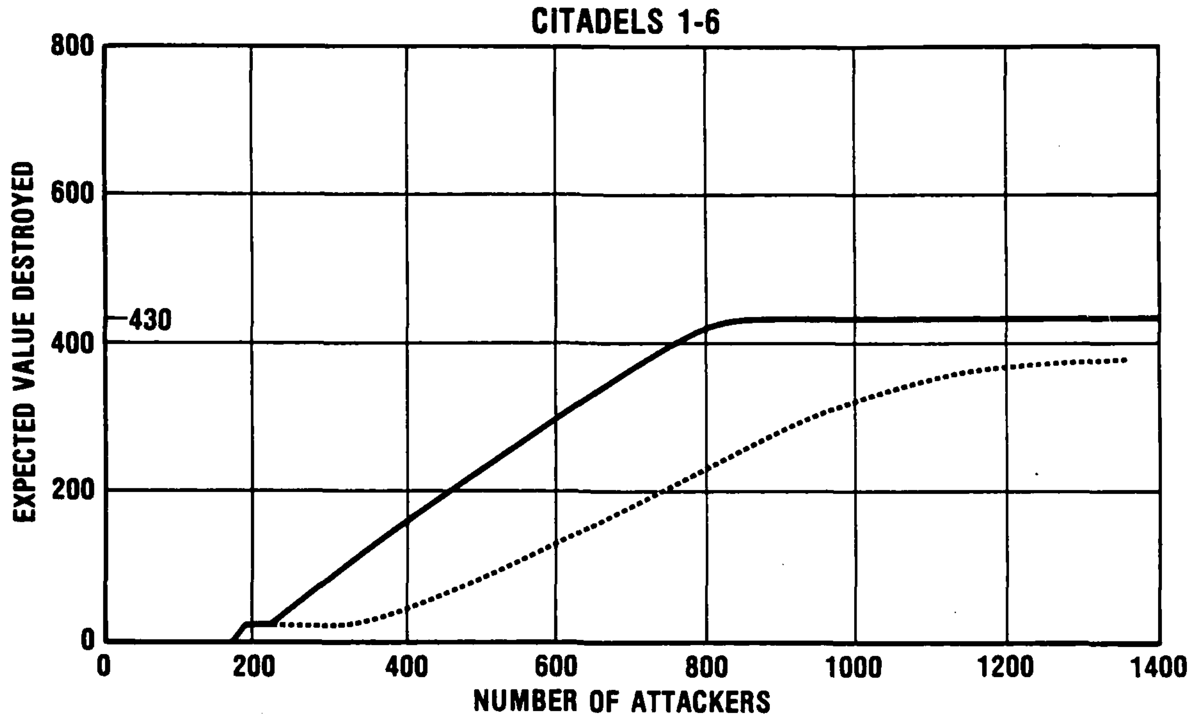


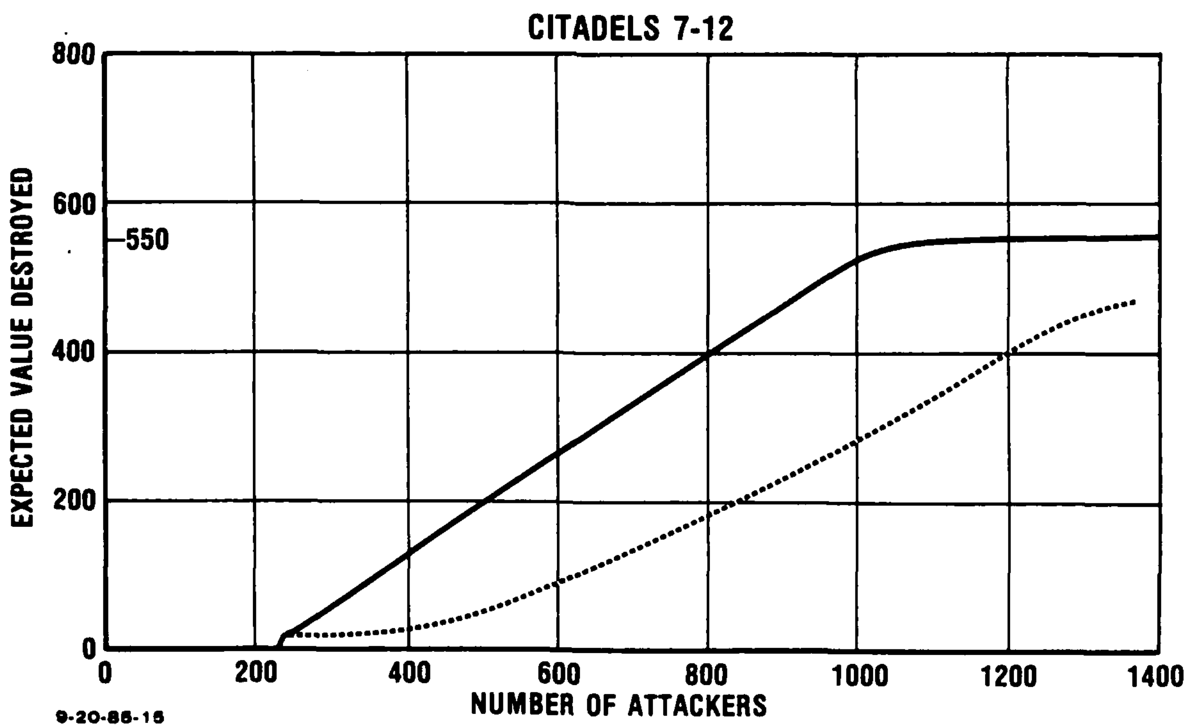
CITADEL 30
780
EXPECTED VALUE DESTROYED
800
600
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS
9-20-85-14

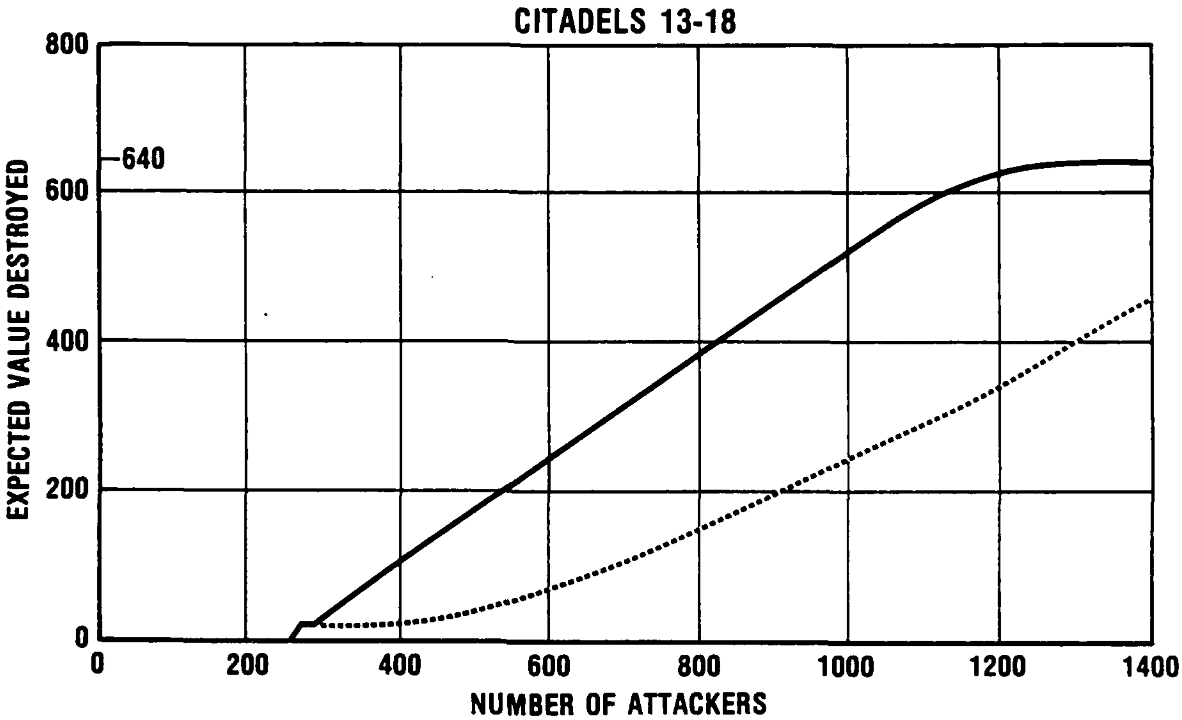
# APPENDIX E

## RESULTS WHEN AREA DEFENSE EQUALS 40 PERCENT OF TARGET VALUE

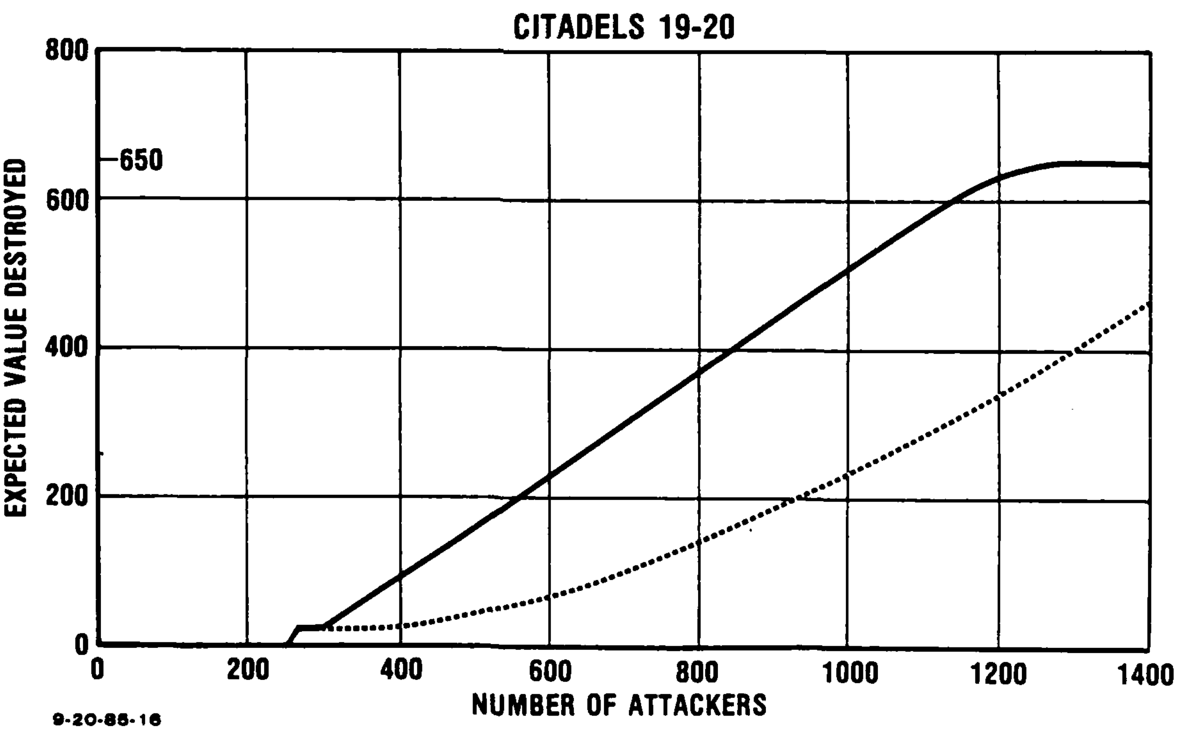
The eleven charts contained in this appendix demonstrate the difference between ipp and no ipp for each of the thirty citadels when the area defense equals forty percent of the target value. On each chart, the solid line represents no ipp and the dashed line represents ipp. The citadels covered by a specific chart are noted at the top of the chart.


CITADELS 1-6
EXPECTED VALUE DESTROYED
800
600
430
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS

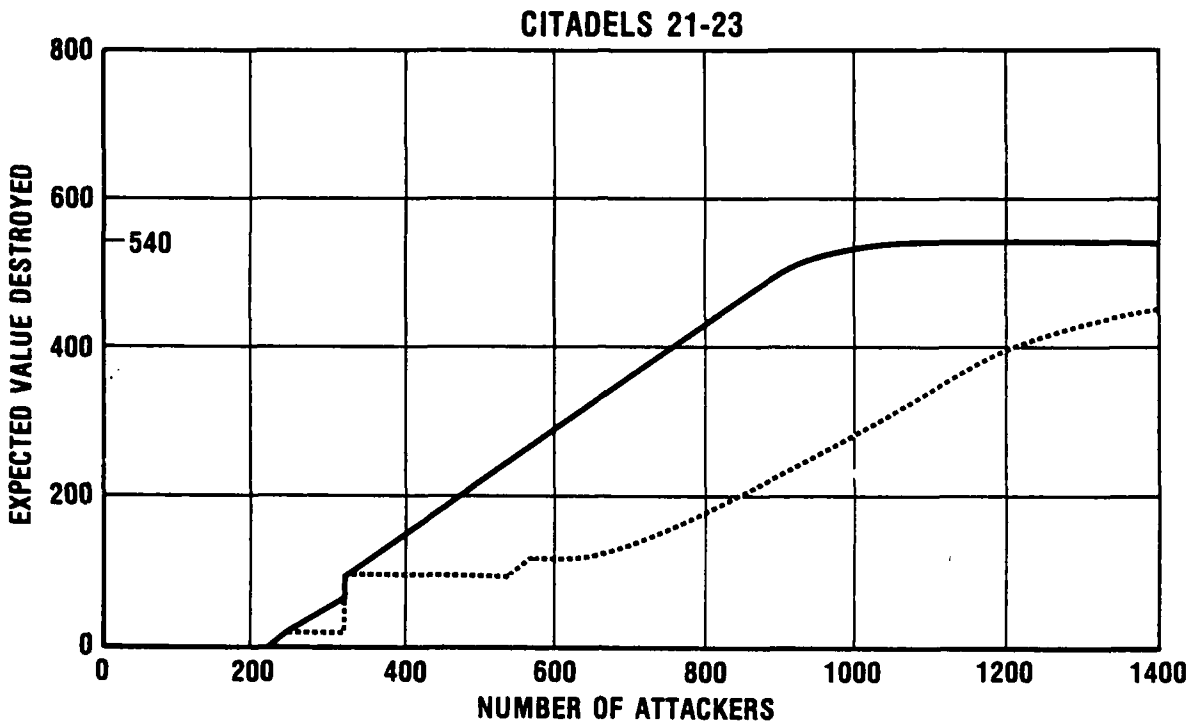

CITADELS 7-12
EXPECTED VALUE DESTROYED
800
600
550
400
200
0
0
200
400
600
800
1000
1200
1400
NUMBER OF ATTACKERS
9-20-86-15

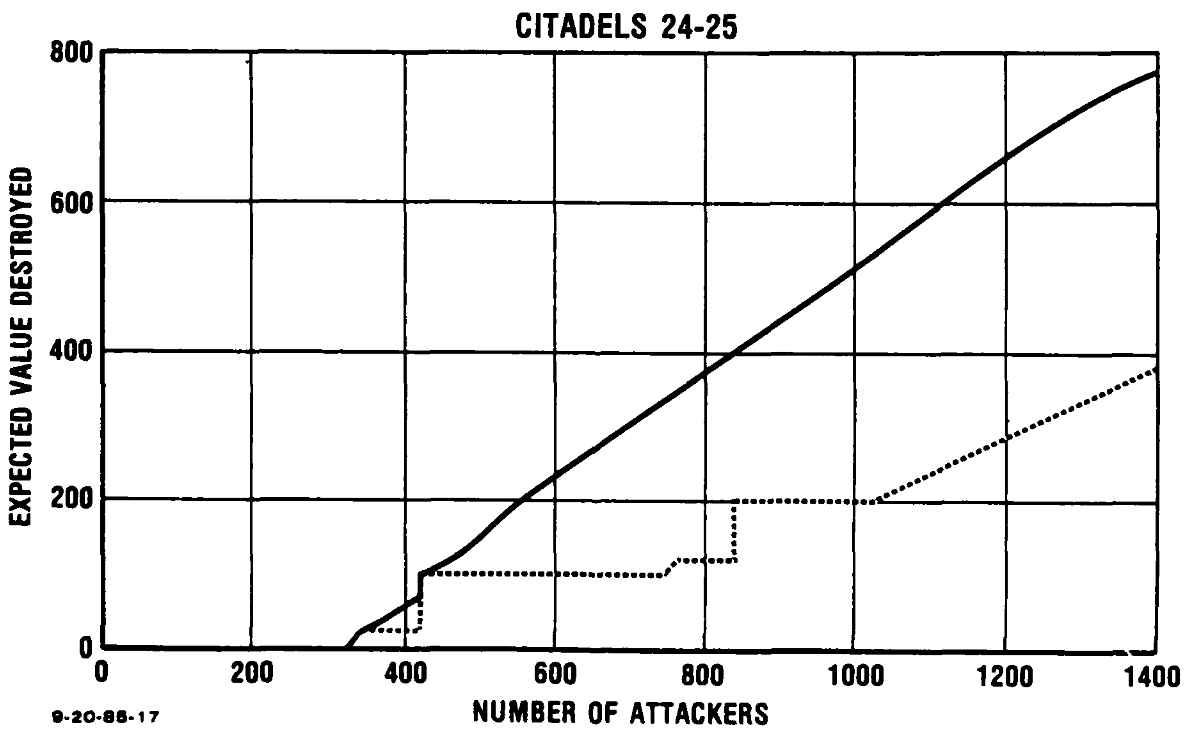


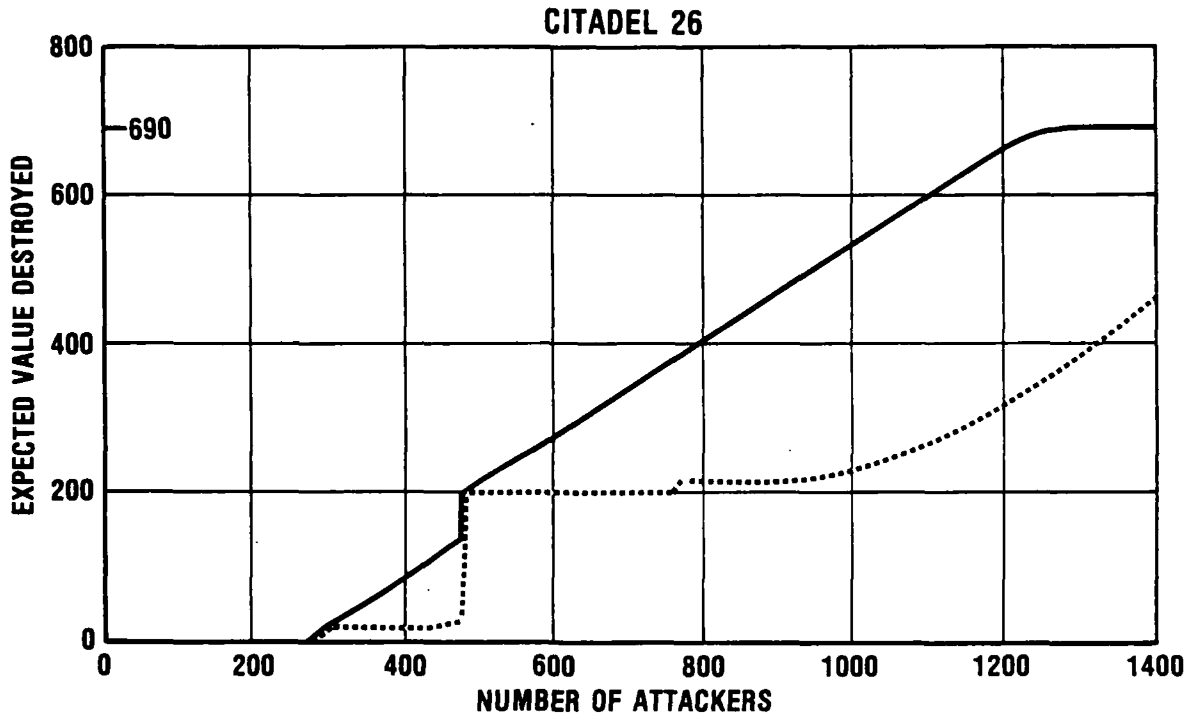
CITADELS 13-18
800
640
600
400
200
0
0
200
400
600
800
1000
1200
1400
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS

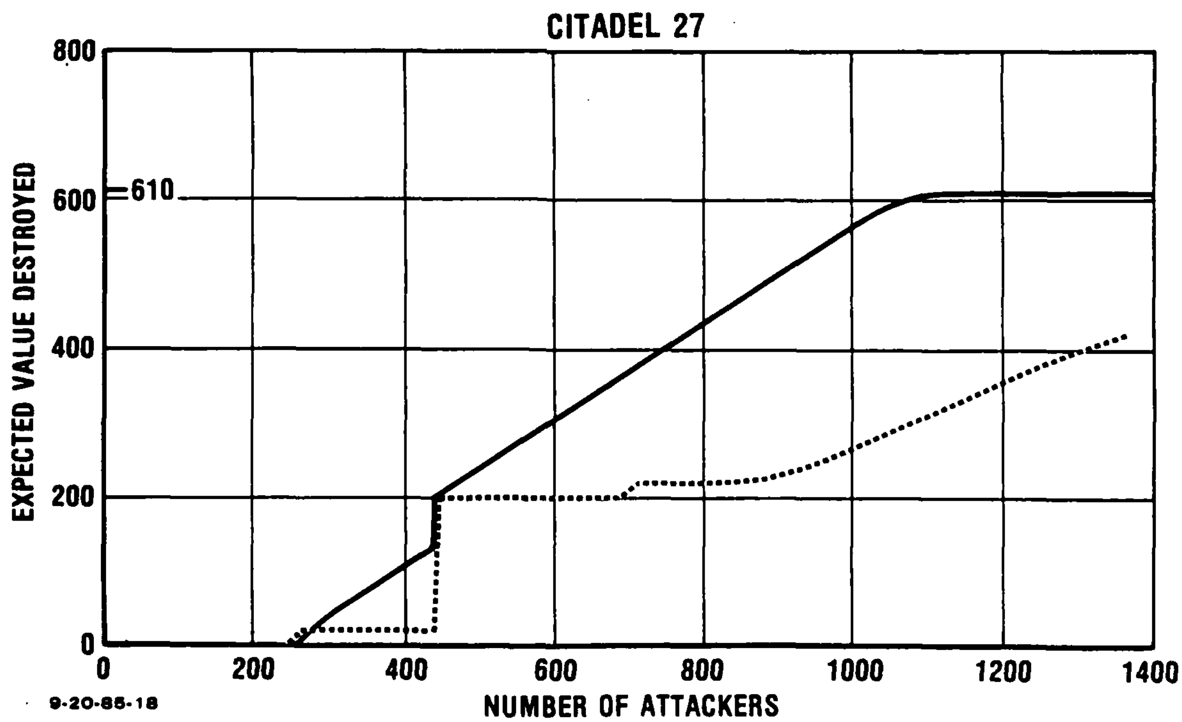



CITADELS 19-20
800
650
600
400
200
0
0
200
400
600
800
1000
1200
1400
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
9-20-85-16


CITADELS 21-23
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
540
0
200
400
600
800
1000
1200
1400

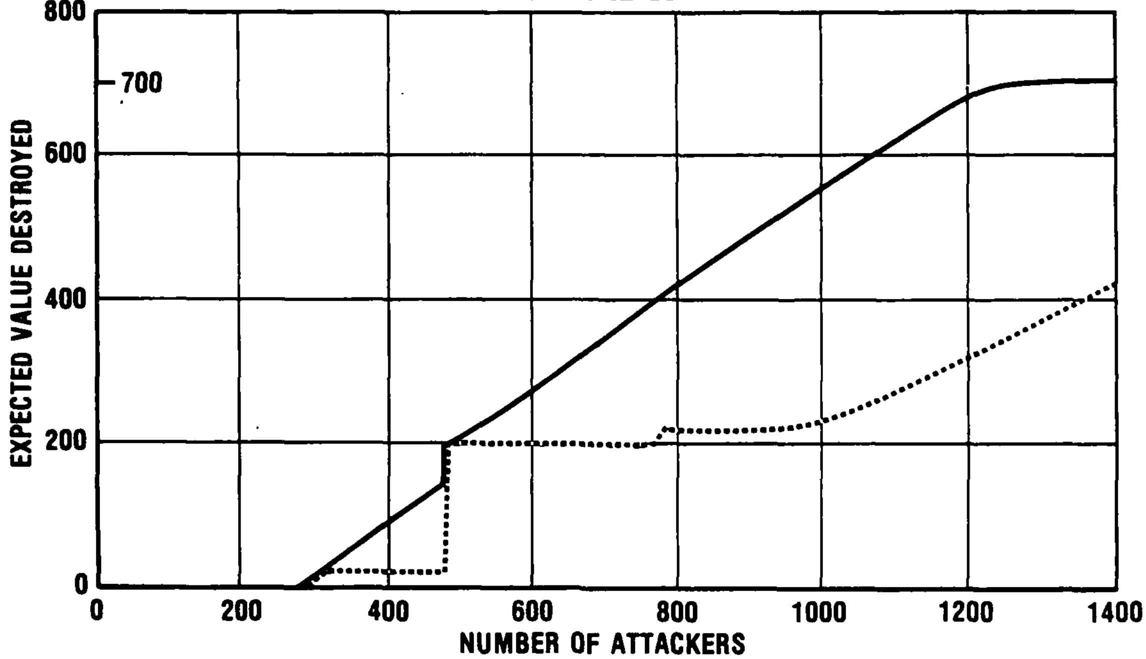

CITADELS 24-25
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
0
200
400
600
800
1000
1200
1400
9-20-86-17


CITADEL 26
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
690
0
200
400
600
800
1000
1200
1400

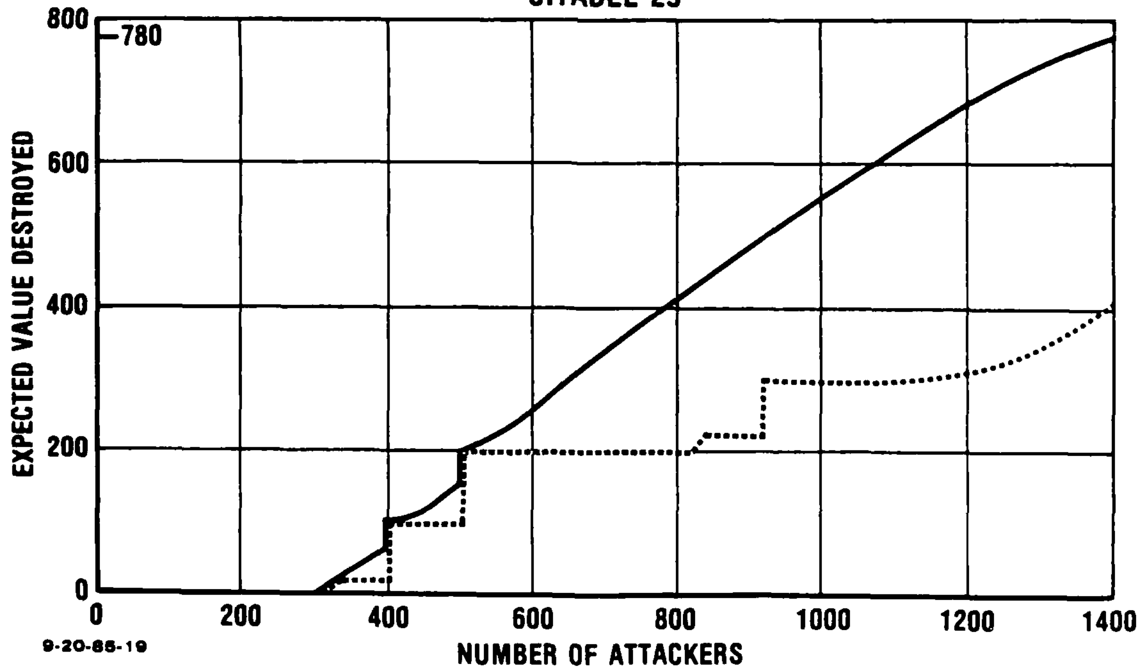

CITADEL 27
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
610
0
200
400
600
800
1000
1200
1400

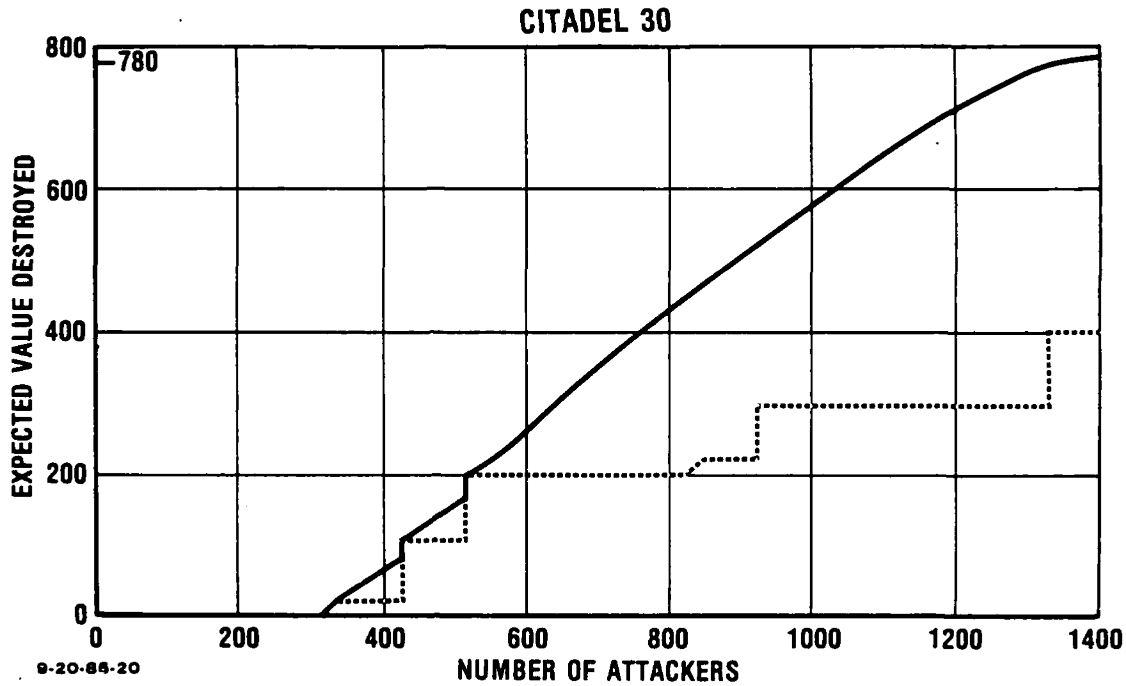
9-20-85-18

## CITADEL 28




## CITADEL 29




CITADEL 30
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
780
0
200
400
600
800
1000
1200
1400
9-20-86-20

# APPENDIX F

## APPROXIMATE RESULTS WHEN AREA DEFENSE EQUALS 100 PERCENT OF TARGET VALUE

The large scale problems considered in this paper in the case of no ipp and area defense = 100% of value presented large computational burdens. To simplify the computations, I derived some approximations which may be applied with benefit.

In order to imvestigate the optimal attacks, I supplemented the FORTRAN program implementing Soland's algorithm (discussed in Appendix A) by the introduction of the variables

attop (i,r) = number of attackers used against target i in an optimal attack when the offense is attacking the first i targets with a total of r RVs,

and

numb (i) = number of attackers used against target i in an optimal attack against the whole target.

The values of attop (i,r) are obtained readily as a byproduct of the dynamic programming algorithm, while the values of numb (i) can be easily obtained from the formulas

$$\text{numb } (T) = \text{attop } (T, a)$$

and

$$\text{numb } (i) = \quad \text{attop } (i, a - \sum_{j=it1}^{T} \text{numb } (j) )$$

for i = T-1, T-2,...,1.

After examining many cases of the optimal attacks, I decided upon two guidelines to aid in the approximations:

1. If the attacker attacks two targets of the same value in his optimal attack, then he attacks them with equal numbers of weapons.
2. When possible, the attacker attacks more valuable targets rather than less valuable targets.

Needless to say, these two guidelines held true in the many cases I examined.

At this point, I assumed that the targets were equal valued and numerous enough that the attacker is not limited in his choice of targets. The arrival probability through the area defense is hypergeometric:

$$\text{Prob}\left\{\begin{array}{l}\text{n RVs arrive,}\\ \text{given that x were}\\ \text{fired at the target}\end{array}\right\} = \frac{\binom{a-b}{n}\binom{b}{x-n}}{\binom{a}{x}}$$

where a = number of attackers and b = number of area defenders. I approximate this hypergeometric distribution by a binomial distribution:

$$\text{Prob}\left\{\begin{array}{l}\text{n RVs arrive,}\\ \text{given that x were}\\ \text{fired at the target}\end{array}\right\} \underline{\underline{\text{approx.}}} \binom{x}{n} p^{x-n} q^{n},$$

where $p = b/a$ and $q = 1 - b/a$.

Now I formulate the optimal attack problem as

$$\underset{x \, \varepsilon \, \{v+1, v+2, \ldots\}}{\text{MAX}} \quad \frac{a}{x} \sum_{n=r+1} \binom{x}{n} (1-b/a)^{n} (b/a)^{x-n}$$

The second term in the product is the Incomplete Beta Function for which I found the following approximation:

$$I_x(a,b) = \sum_{n=a}^{a+b-1} \binom{a+b-1}{n} x^n (1-x)^{a+b-1-n}$$

$$\cong P(y) + \varepsilon,$$

where

$$|\varepsilon| < 5 \times 10^{-3} \text{ if } a + b > 6 \text{ and } (a + b - 1)(1 - x) \leq .8$$

and

$$y = \frac{3\left[w_1\left(1-\frac{1}{9b}\right) - w_2\left(1-\frac{1}{9a}\right)\right]}{\left(\frac{w_1^2}{b} + \frac{w_2^2}{a}\right)^{1/2}}$$

$$w_1 = (bx)^{1/3}$$

$$w_2 = (a(1-x))^{1/3}$$

$$P(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-1/2\, t^2}\, dt \ .$$

Furthermore, P(y) has the following rational approximation:

$$P(y) = 1 - 1/2\,(1 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4)^{-4} + \varepsilon$$

where

$$|\varepsilon| < 2.5 \times 10^{-4}$$

and

$$a_1 = .196854$$
$$a_2 = .115194$$
$$a_3 = .000344$$
$$a_4 = .019527 \ .$$

If these two approximations are used on $I_q$ (v + 1, x − v), then we can numerically maximize the product

$$\frac{a}{x} I_q(v+1, x-v)$$

for x greater than v, assuming the values of v and b/a to be given.

The results are plotted in Figure F-1 which shows β vs. b/a for different values of v. The quantity β is defined by

$$\beta = 100 \cdot \left(\frac{\text{optimal RVs/target}}{(v/q)} - 1\right).$$

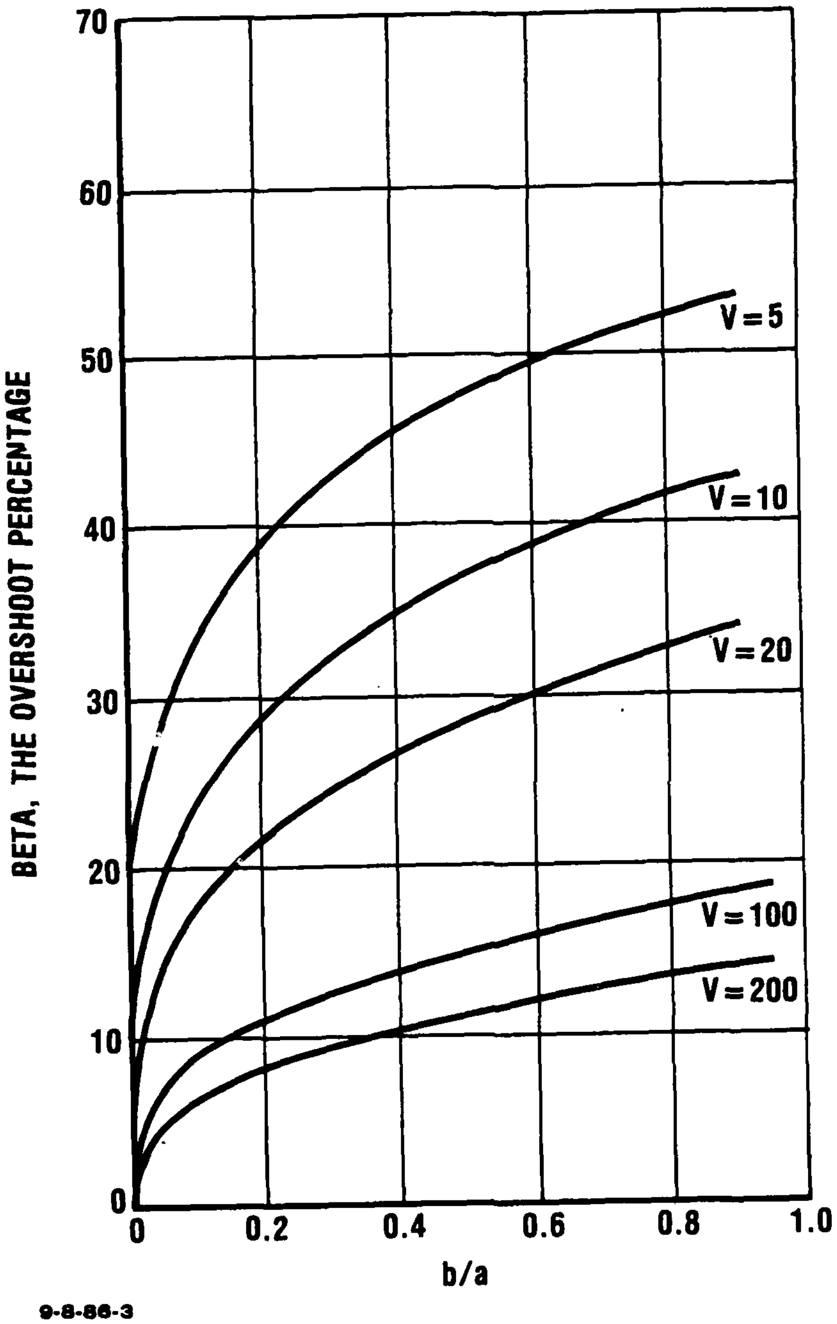



Figure F-1. Optimal Overshoot Percentages

Figure F-2 shows the probability of kill in the optimal attack. Together these two charts are the basis of finding the approximate optimal attack. As an example, consider citadel 29 and 1400 attackers. Using Figure F-1, we find that

| Value | Optimal Number of Attackers | $P_k$ |
|---|---|---|
| 200 | 503 | .9786 |
| 100 | 260 | .9685 |
| 20 | 58 | .9213 |
| 5 | 17 | .8286. |

With 1400 weapons, the attacker will use 503 against the 200 value target, 260 against the 100 value, 58 against each of the 4 targets of value 20 and 31 against 13 targets of value 10. Two RVs will be left over. Summing up the expected damages at each target, we find that the total expected damage is 481.10.

This method was followed to generate the no ipp curves for each of the individual citadels. The ipp curves were generated as before with a combination of shoot-t0-kill and defense dilution strategies. The individual citadels were then integrated into a nationwide system by using dynamic programming.

Table F-1 summarizes the differences of RVs required with no impact point prediction and with impact point prediction in selected cases. Table F-2 gives numerical details, and Figure F-3 displays the aggregate information. The remaining figures display results for citadels in the same format as in Appendix D and E for the 10 percent and 40 percent cases.

**Table F-1. AREA DEFENSE = 100% OF VALUE**

| | Number of Attackers | | | |
|---|---|---|---|---|
| Value | No ipp | ipp | Difference | Percentage Difference |
| 4,450 (1/4) | 11,100 | 24,805 | 13,750 | 123.87% |
| 8,900 (1/2) | 22,600 | 50,800 | 28,200 | 124.78% |
| 13,350 (3/4) | 34,250 | 84,950 | 50,700 | 148.03% |
| 17,800 (All) | 47,050 | | | |

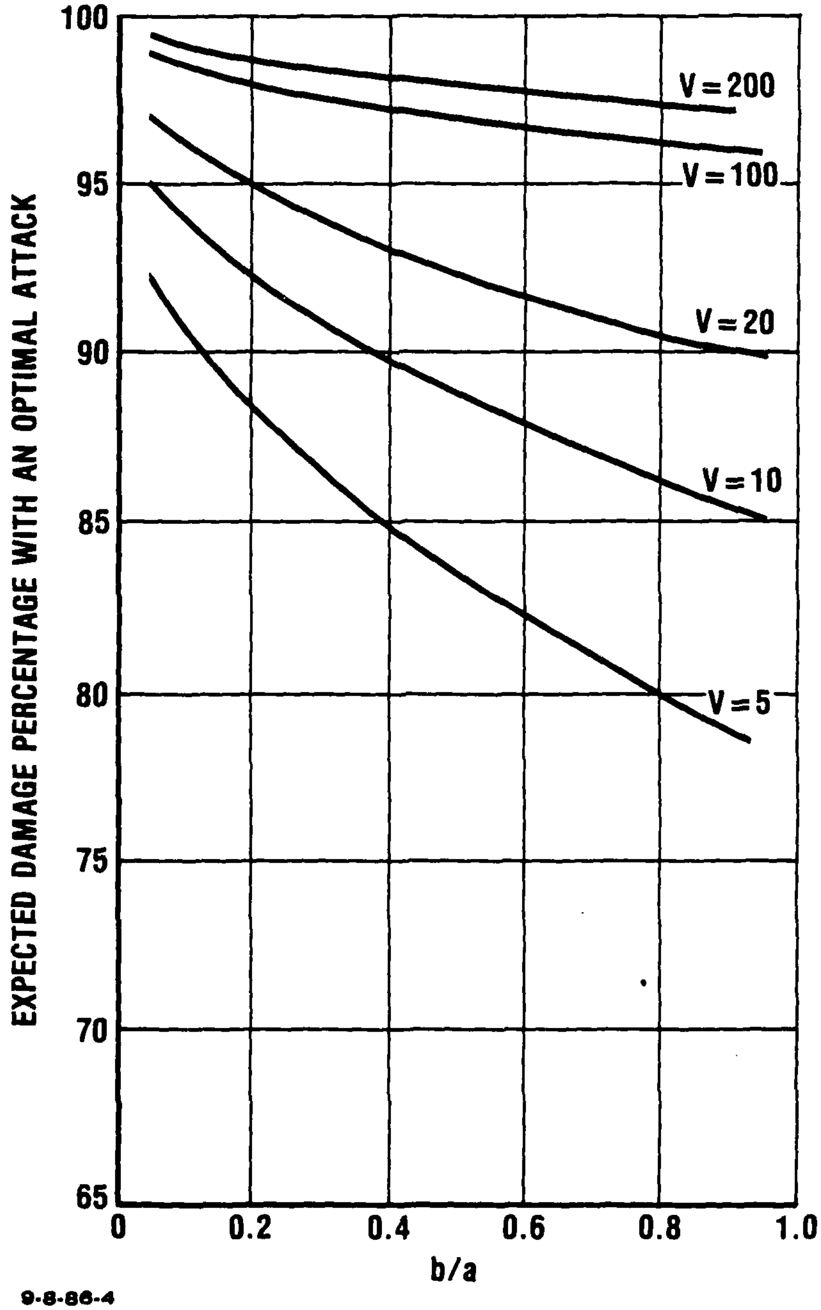



Figure F-2. Expected Damage Percentage with an Optimal Attack.

**Table F-2. TARGET SET OF 30 CITADELS**
Area Defense = 100 percent of value

| Attackers | no ipp | ipp |
|---|---|---|
| 1000 | 372 | 200 |
| 2000 | 784 | 400 |
| 3000 | 1214 | 604.5122 |
| 4000 | 1591 | 817.4713 |
| 5000 | 2042 | 1017.471 |
| 6000 | 2425 | 1217.471 |
| 7000 | 2836 | 1393.939 |
| 8000 | 3230 | 1566.130 |
| 9000 | 3622 | 1737.484 |
| 10000 | 4023 | 1909.048 |
| 11000 | 4412 | 2080.798 |
| 12000 | 4813 | 2252.357 |
| 13000 | 5202 | 2424.009 |
| 14000 | 5586 | 2595.466 |
| 15000 | 5992 | 2868.117 |
| 16000 | 6377 | 2938.561 |
| 17000 | 6756 | 3110.201 |
| 18000 | 7151 | 3281.761 |
| 19000 | 7534 | 3453.333 |
| 20000 | 7917 | 3624.869 |
| 21000 | 8299 | 3796.405 |
| 22000 | 8681 | 3967.983 |
| 23000 | 9064 | 4139.537 |
| 24000 | 9447 | 4311.188 |
| 25000 | 9830 | 4482.627 |
| 26000 | 10213 | 4654.273 |
| 27000 | 10595 | 4825.741 |
| 28000 | 10978 | 4997.365 |
| 29000 | 11361 | 5168.940 |
| 30000 | 11741 | 5340.476 |
| 31000 | 12124 | 5512.001 |
| 32000 | 12507 | 5683.561 |
| 33000 | 12890 | 5855.146 |
| 34000 | 13273 | 6026.652 |
| 35000 | 13645 | 6198.212 |

Table F-2 (Continued)

| Attackers | no ipp | ipp |
|---|---|---|
| 36000 | 14028 | 6369.749 |
| 37000 | 14411 | 6541.273 |
| 38000 | 14781 | 6712.849 |
| 39000 | 15150 | 6884.418 |
| 40000 | 15536 | 7055.943 |
| 41000 | 15913 | 7227.484 |
| 42000 | 16280 | 7399.020 |
| 43000 | 16658 | 7570.545 |
| 44000 | 17041 | 7742.121 |
| 45000 | 17349 | 7913.646 |
| 46000 | 17600 | 8058.171 |
| 47000 | 17795 | 8256.610 |
| 48000 | 17800 | 8428.137 |
| 49000 | 17800 | 8599.755 |
| 50000 | 17800 | 8771.280 |
| 51000 | 17800 | 8942.657 |
| 52000 | 17800 | 9114.442 |
| 53000 | 17800 | 9285.490 |
| 54000 | 17800 | 9455.623 |
| 55000 | 17800 | 9623.533 |
| 56000 | 17800 | 9790.361 |
| 57000 | 17800 | 9947.421 |
| 58000 | 17800 | 10109.54 |
| 59000 | 17800 | 10268.95 |
| 60000 | 17800 | 10426.43 |
| 61000 | 17800 | 10587.74 |
| 62000 | 17800 | 10740.23 |
| 63000 | 17800 | 10897.48 |
| 64000 | 17800 | 11050.45 |
| 65000 | 17800 | 11207.02 |
| 66000 | 17800 | 11356.84 |
| 67000 | 17800 | 11503.34 |
| 68000 | 17800 | 11649.84 |
| 69000 | 17800 | 11791.78 |
| 70000 | 17800 | 11926.73 |

Continued

Table F-2 (Continued)

| Attackers | no ipp | ipp |
|---|---|---|
| 71000 | 17800 | 12055.48 |
| 72000 | 17800 | 12177.28 |
| 73000 | 17800 | 12293.63 |
| 74000 | 17800 | 12404.39 |
| 75000 | 17800 | 12510.41 |
| 76000 | 17800 | 12611.87 |
| 77000 | 17800 | 12708.53 |
| 78000 | 17800 | 12801.39 |
| 79000 | 17800 | 12890.34 |
| 80000 | 17800 | 12975.64 |
| 81000 | 17800 | 13057.68 |
| 82000 | 17800 | 13136.09 |
| 83000 | 17800 | 13212.00 |
| 84000 | 17800 | 13284.88 |
| 85000 | 17800 | 13355.02 |
| 86000 | 17800 | 13422.77 |
| 87000 | 17800 | 13488.64 |
| 88000 | 17800 | 13554.51 |
| 89000 | 17800 | 13620.38 |
| 90000 | 17800 | 13684.86 |
| 91000 | 17800 | 13747.02 |
| 92000 | 17800 | 13807.03 |
| 93000 | 17800 | 13864.74 |
| 94000 | 17800 | 13920.50 |
| 95000 | 17800 | 13974.53 |
| 96000 | 17800 | 14026.69 |
| 97000 | 17800 | 14077.42 |
| 98000 | 17800 | 14126.52 |
| 99000 | 17800 | 14174.13 |
| 100000 | 17800 | 14220.42 |

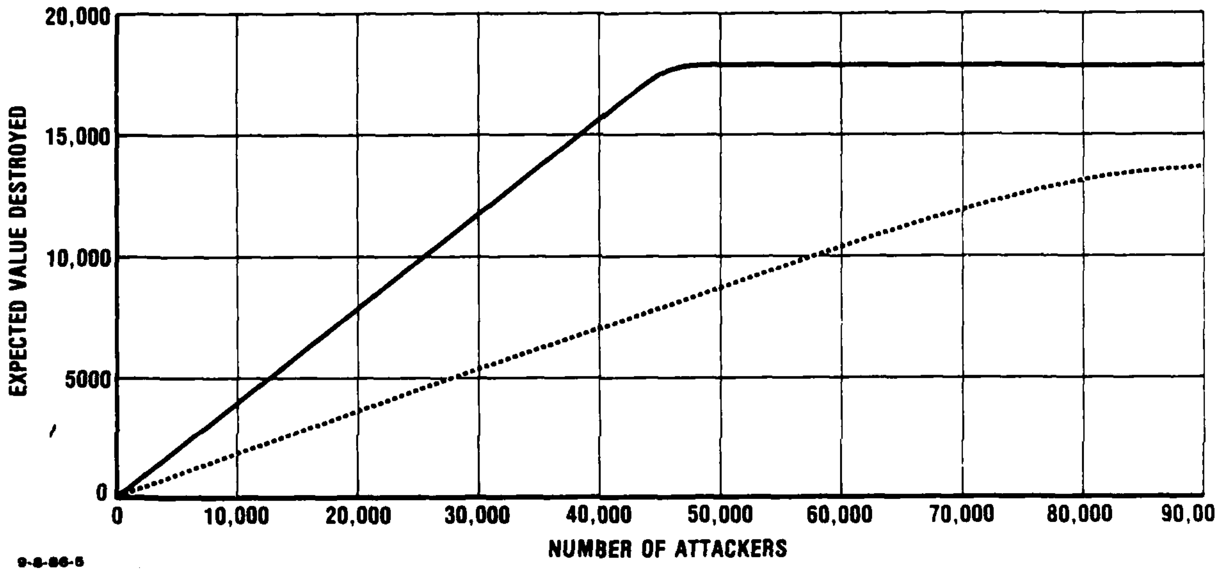
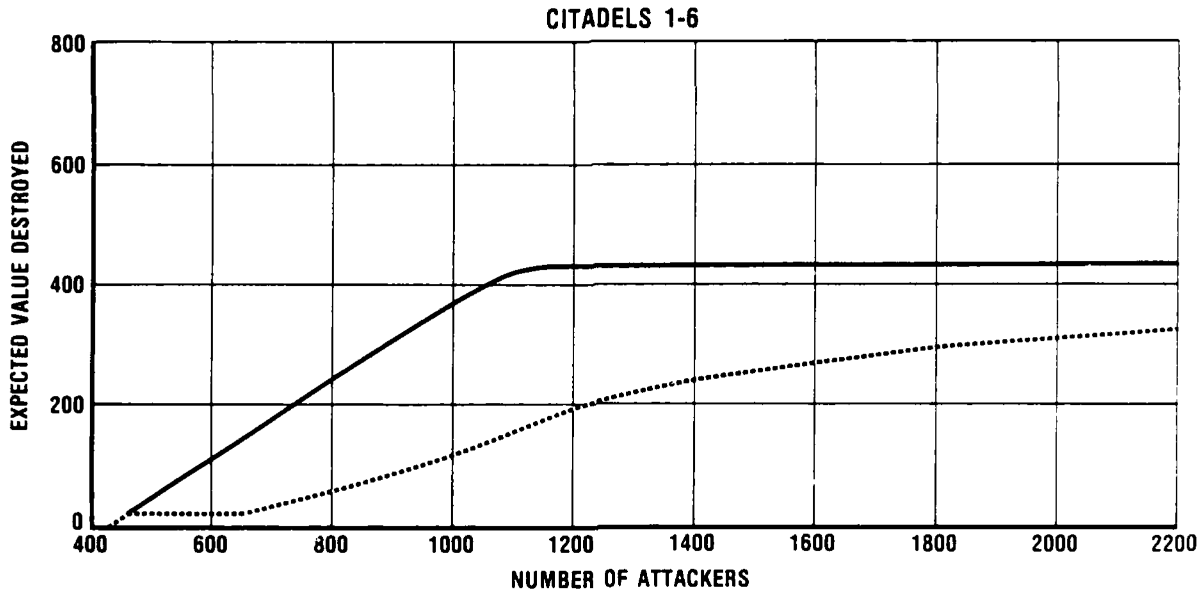
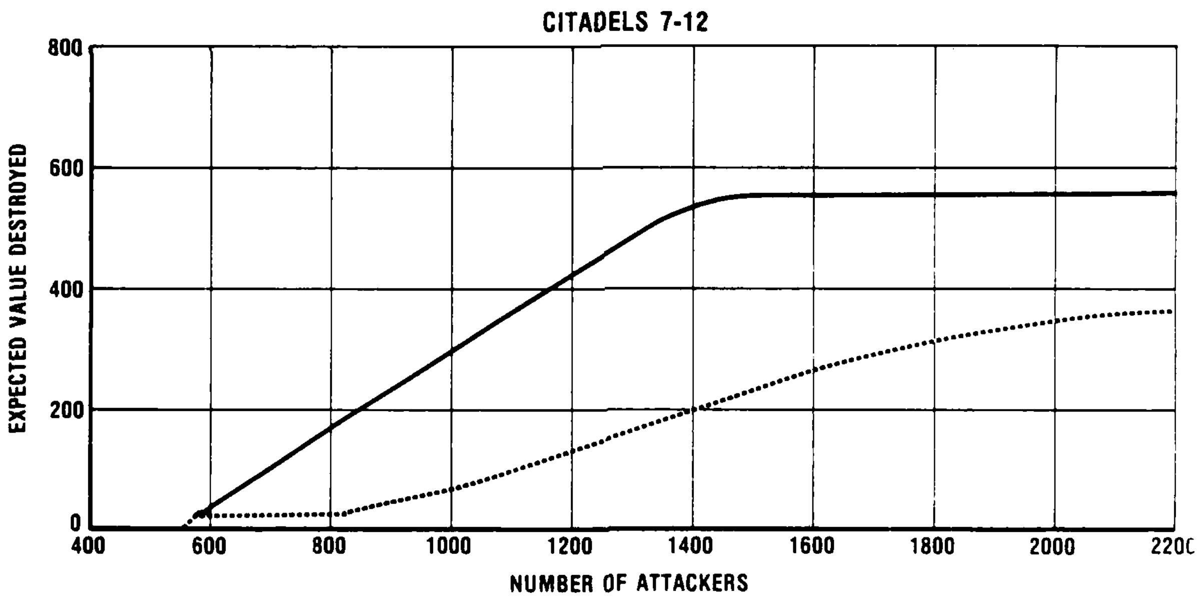



Figure F-3. Difference between ipp and no ipp for a representative nationwide target set when the area defense is equal to 100 percent of the target value.



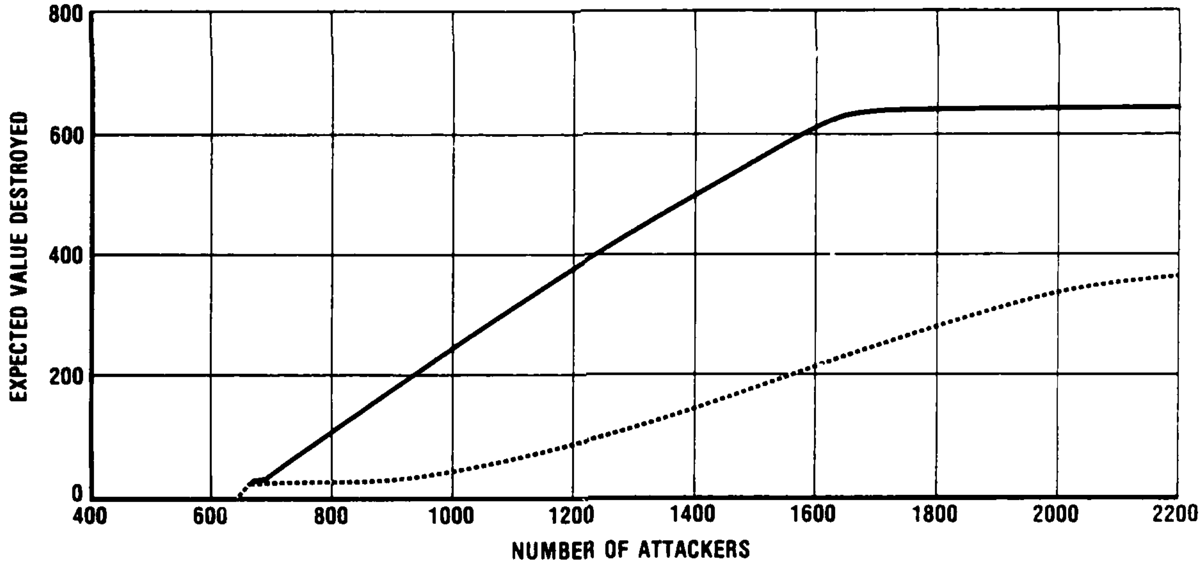
CITADELS 1-6
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
800
600
400
200
0
400
600
800
1000
1200
1400
1600
1800
2000
2200

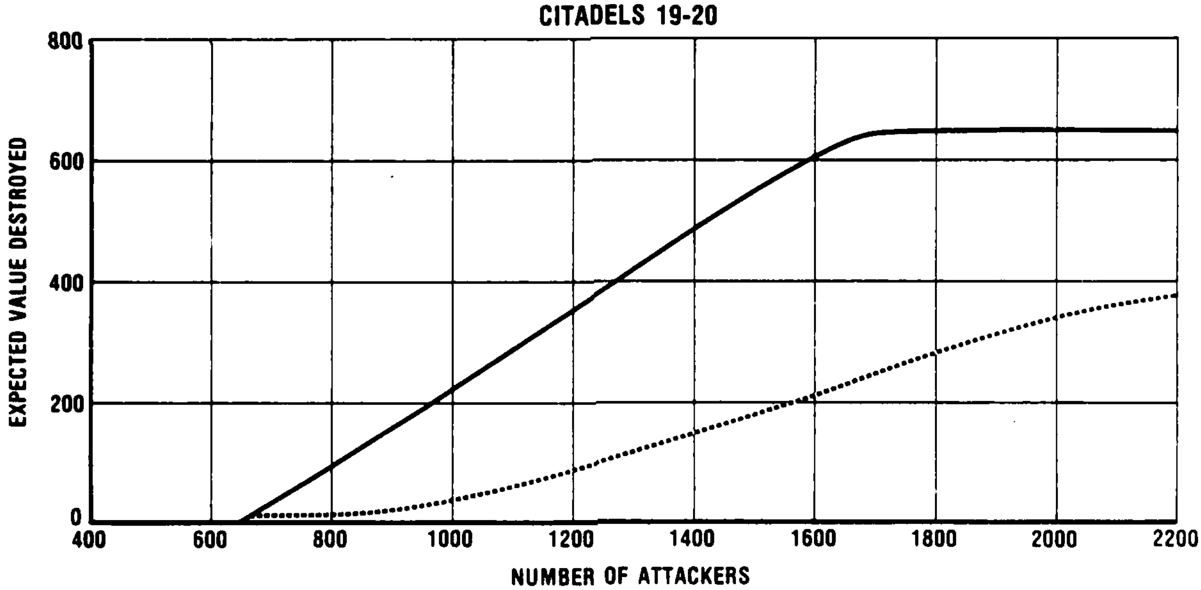
CITADELS 7-12
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS
800
600
400
200
0
400
600
800
1000
1200
1400
1600
1800
2000
2200


9-8-86-6

CITADELS 13-18

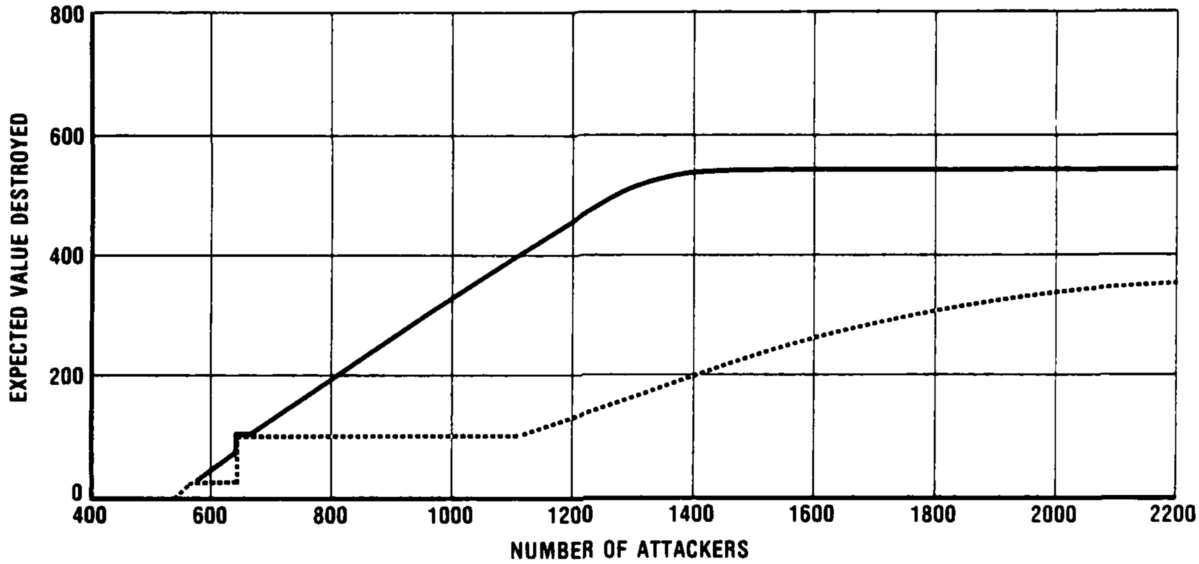



CITADELS 19-20

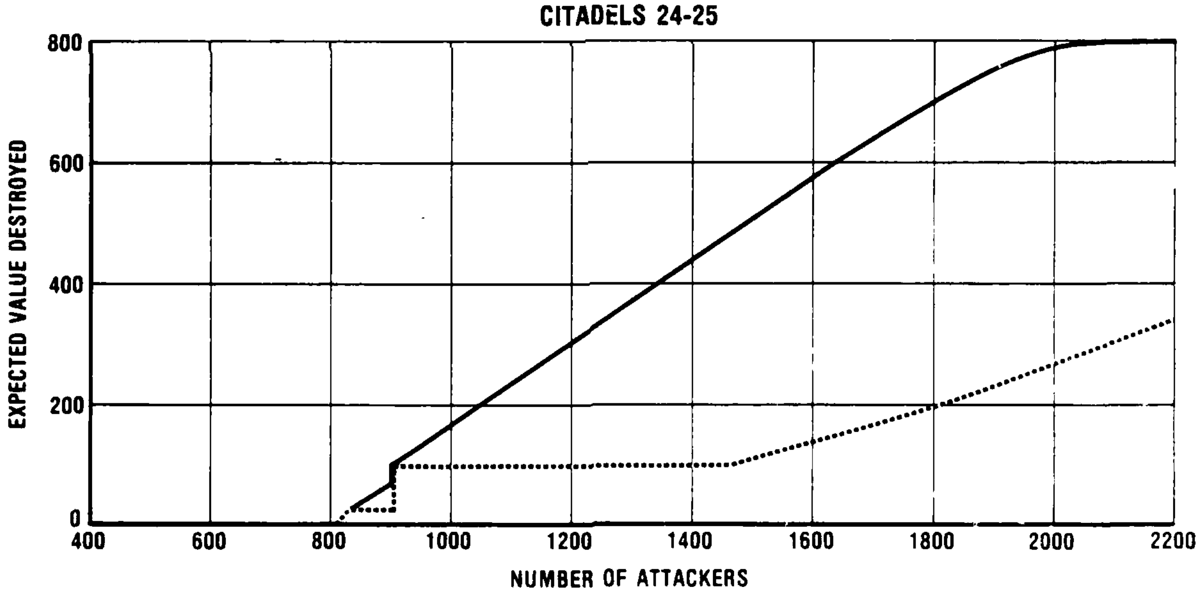



9-8-86-7

CITADELS 21-23




CITADELS 24-25




9-6-86-6

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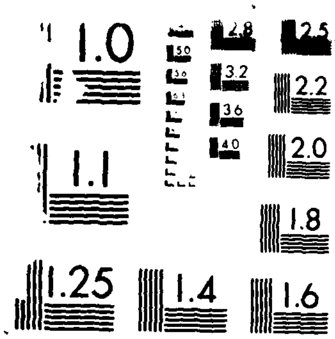
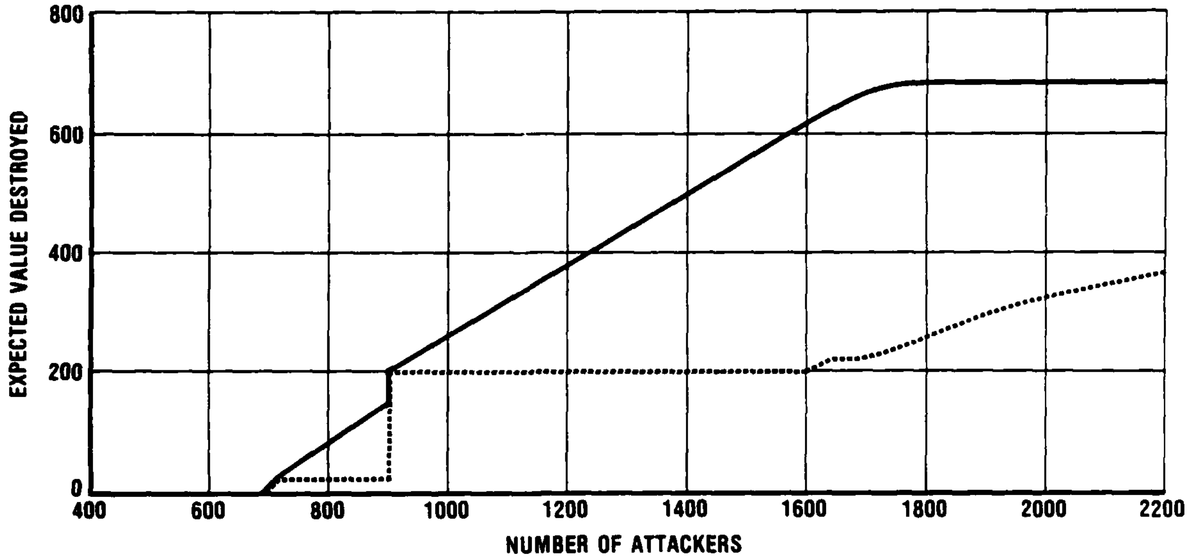
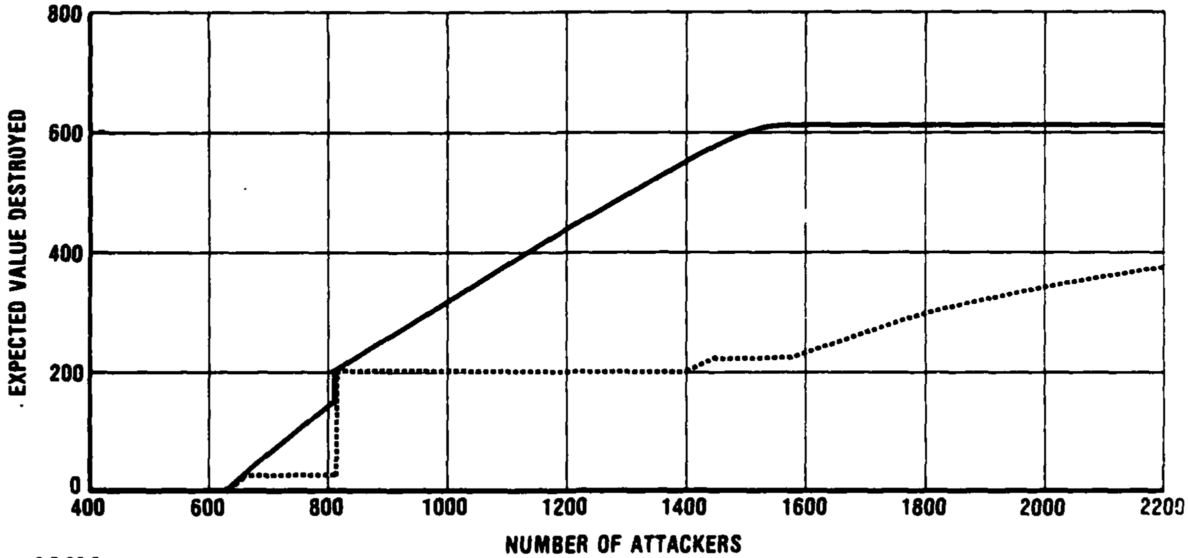


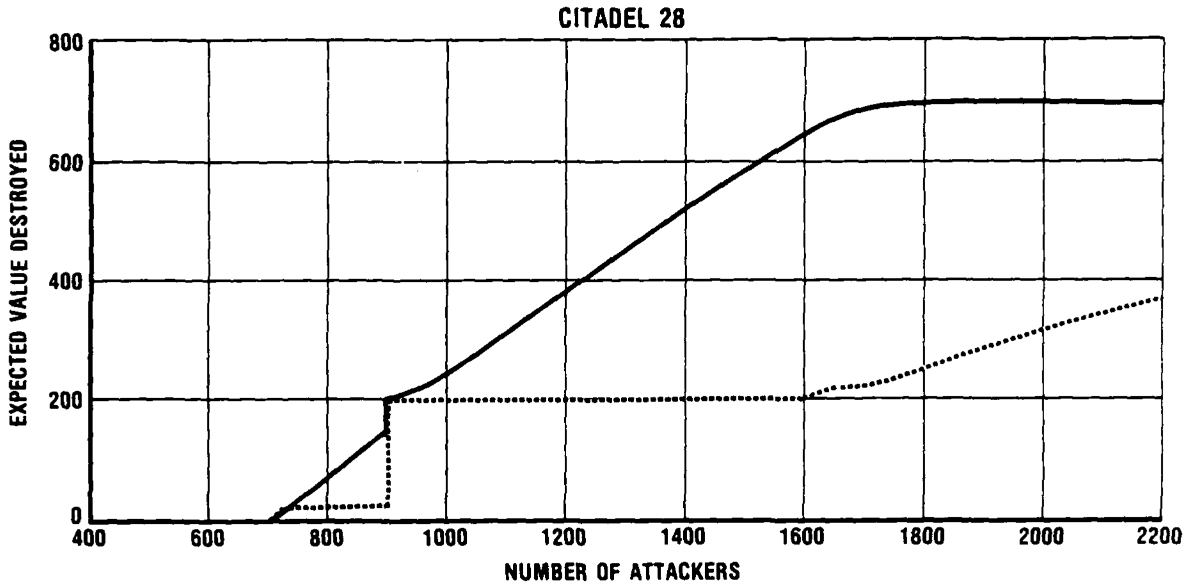

RESOLUTION TEST CHART

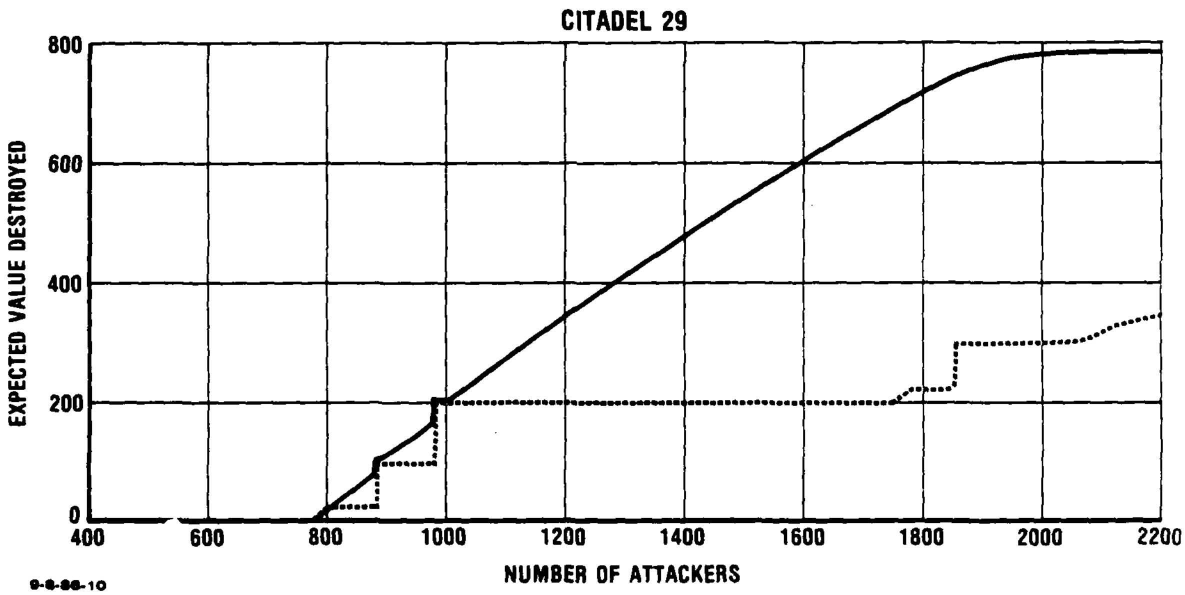
CITADEL 26

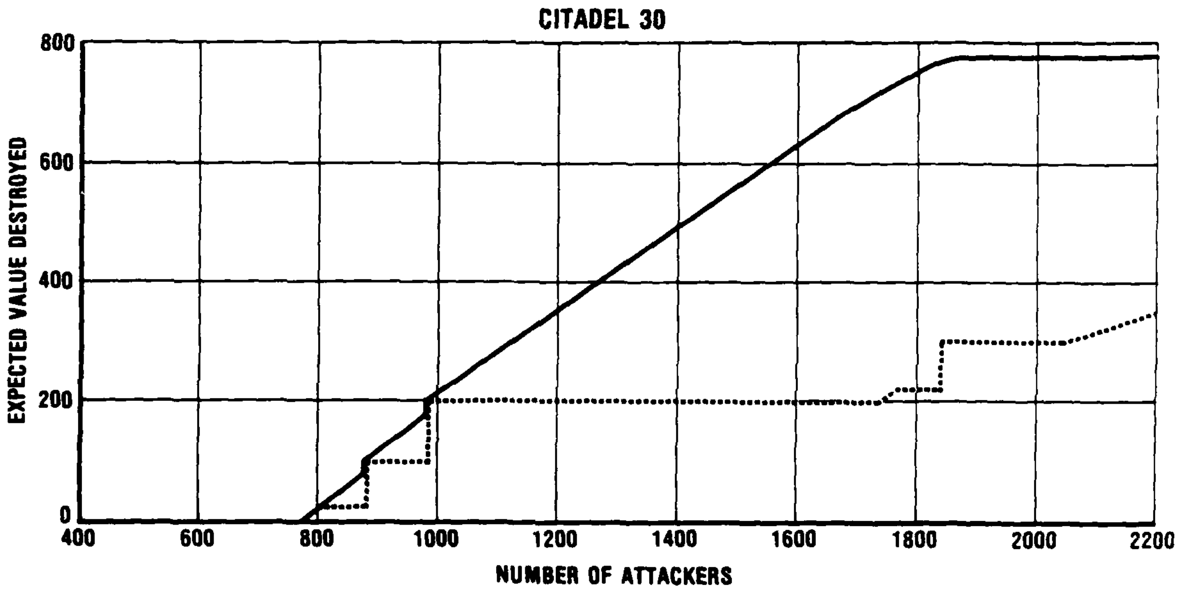



CITADEL 27




CITADEL 28
EXPECTED VALUE DESTROYED
0
200
400
600
800
400
600
800
1000
1200
1400
1600
1800
2000
2200
NUMBER OF ATTACKERS


CITADEL 29
EXPECTED VALUE DESTROYED
0
200
400
600
800
400
600
800
1000
1200
1400
1600
1800
2000
2200
NUMBER OF ATTACKERS
9-8-86-10


CITADEL 30
800
600
400
200
0
400
600
800
1000
1200
1400
1600
1800
2000
2200
EXPECTED VALUE DESTROYED
NUMBER OF ATTACKERS

9-8-86-11

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|---|---|
| Dr. J. Bracken | 2 |
| Dr. W.J. Schultis | 1 |
| Mr. R.B. Pirie | 1 |
| Mr. S.J. Deitchman | 1 |
| Control & Distribution | 4 |

END

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